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A WAY TO SIMPLIFY TRUTH FUNCTIONS

W. V. QUINE, Harvard University

The quest is herewith resumed of a convenient technique for converting a truth-functional formula into its shortest equivalent in alternational normal form. We may, as before,* confine our attention to formulas given in alternational normal form. These may be described as comprising all *literals* (letters and negations of letters), and more generally all *fundamental formulas* (literals and conjunctions of literals, containing no letter twice), and more generally all alternations of fundamental formulas.

It will also be convenient, for purposes of the technique here to be developed, to exclude from consideration those formulas which are valid, or tautologous. A formula can be quickly tested for validity, and if found valid it can be rewritten in simplest form, '$p \vee \bar{p}$', out of hand. So let us assume hereafter that the formulas under investigation are in alternational normal form but not valid.

Where ϕ and ψ are fundamental formulas, we say that ϕ *subsumes* ψ if all the literals whereof ψ is a conjunction are among the literals whereof ϕ is a conjunction. We call ϕ a *prime implicant* of a formula Φ if ϕ implies Φ and subsumes no shorter formula which implies Φ. Now any shortest equivalent (in alternational normal form, as usual) of a formula Φ is an alternation of prime implicants of Φ (PSTF, p. 524); so a major part of the job of finding a shortest equivalent of Φ is the eliciting of all the prime implicants of Φ. A drawback of the procedure in PSTF, there remarked upon (p. 531), was that the prime implicants were exhausted only with help of a preliminary expansion into the cumbersome "developed normal form." Now, on the other hand, a speedy and direct method will be explained for getting all the prime implicants. Φ can be transformed into the alternation of all its prime implicants simply by continued use of the following operations (i) and (ii).**

(i) *Drop these obvious superfluities:* If one of the clauses of alternation subsumes another, drop the subsuming clause. Also supplant $\alpha \vee \bar{\alpha}\phi$ by $\alpha \vee \phi$ (and $\bar{\alpha} \vee \alpha\phi$ by $\bar{\alpha} \vee \phi$), where α is a single letter.

(ii) *Adjoin, as an additional clause of alternation, the* consensus *of two clauses.* Definition: The conjunction $\phi\psi$ (with any duplicate literals deleted) is called the *consensus* of $\alpha\phi$ and $\bar{\alpha}\psi$, provided that it contains no letter both affirmed and negated. The operation (ii) is to be regarded as not applying in case the

* *The problem of simplifying truth functions*, this MONTHLY, vol. 59, 1952, pp. 521–531; cited hereafter as PSTF, but not drastically presupposed.

** *Note added June 7, 1955:* It has today come to my attention, more than two months after submission of the present paper, that this result was anticipated in an Air Force memorandum of April 1954 by Edward W. Samson and Burton E. Mills, *Circuit minimization: algebra and algorithms for new Boolean canonical expressions*, AFCRC Technical Report 54-21. For my present paper I would still plead brevity and perspicuity, and certain novelties in the later portion; but to Samson and Mills belongs the credit for discovering that the alternation of prime implicants can be got by (i) and (ii).

consensus subsumes a clause already present; otherwise we could get an unending oscillation of (i) and (ii).

The two operations are to be performed as long as possible. (i), in particular, is to be performed as much as possible before and after each performance of (ii). When neither is applicable further, then, as will be proved, we have the alternation of all and only the prime implicants. First let us see an example.

(1) $ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee prs \vee qr\bar{s} \vee pqrt.$

By (i) we drop 'prs', which subsumes 'ps'. By (ii), next, we add on the consensus 'pqr' of 'ps' and '$qr\bar{s}$'. Result:

(2) $ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee qr\bar{s} \vee pqrt \vee pqr.$

By (i) now we drop '$pqrt$', which subsumes 'pqr'. Result:

(3) $ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee qr\bar{s} \vee pqr.$

By (ii), finally, we add on the consensus '$r\bar{s}t$' of '$\bar{q}t$' and '$qr\bar{s}$', and also the consensus 'prt' of '$\bar{q}t$' and 'pqr', obtaining:

(4) $ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee qr\bar{s} \vee pqr \vee r\bar{s}t \vee prt.$

Here the process ends. No further pair of clauses has a consensus, except such as would subsume (indeed match) an existing clause.

Now to the proof that (i) and (ii) deliver all the prime implicants. This will be proved by proving that any (non-valid) formula Φ is still susceptible to an application of (i) or (ii) as long as there is a prime implicant χ of Φ which is not a clause of Φ.

Φ is implied by χ, yet not valid; so χ must have letters in common with Φ. Moreover, since χ is a *prime* implicant, it has no letters foreign to Φ; for, any such could be dropped without impairing the implication.* Moreover, since χ is a prime implicant of Φ, and each clause of Φ also implies Φ, no clause of Φ other than χ itself is subsumed by χ; hence none, since χ is not a clause of Φ. So there is at least one fundamental formula (χ itself, for one) fulfilling these three conditions: (a) it subsumes χ, (b) it subsumes no clause of Φ, and (c) it contains only letters of Φ. Let ψ be a longest fundamental formula fulfilling (a), (b), and (c). Still ψ will lack some letter α of Φ. (For, if ψ contained all letters of Φ, then, by (b), ψ would conflict with each clause of Φ in point of the affirming or negating of some letter or other; whereas we know rather, by (a), that ψ implies Φ.) Now since ψ is a longest formula fulfilling (a), (b), and (c), the longer formulas $\alpha\psi$ and $\bar{\alpha}\psi$ must fail to fulfill (b); for they do fulfill (a) and (c). So $\alpha\psi$ and $\bar{\alpha}\psi$ each subsumes a clause of Φ. These subsumed clauses must contain α and $\bar{\alpha}$ respectively, since they were not subsumed by ψ alone. But these clauses

* The reasoning in this sentence and the preceding one is an improvement on the proof of Theorem 2 in PSTF, where the case of prime implicants consisting of a single literal was overlooked. Strictly speaking, Theorem 2 fails for tautologies, since any literal is by definition a prime implicant of any tautology.

are not simply α and $\bar{\alpha}$, or Φ would be valid. So there are just three possible cases: the clauses are respectively $\alpha\phi$ and $\bar{\alpha}$, where ψ subsumes ϕ (Case 1); or they are α and $\bar{\alpha}\phi'$, where ψ subsumes ϕ' (Case 2); or they are $\alpha\phi$ and $\bar{\alpha}\phi'$ (Case 3). In Case 1, however, Φ contains $\bar{\alpha} \vee \alpha\phi$ and is accordingly susceptible to operation (i). Similarly for Case 2. In Case 3, finally, Φ is susceptible to an application of (ii), consisting in the adding on of the consensus of $\alpha\phi$ and $\bar{\alpha}\phi'$. This consensus, namely $\phi\phi'$ (minus any duplicate literals), is readily seen to meet the requirements of (ii): it contains no letter both affirmed and negated, since it is subsumed by a fundamental formula ψ; and it subsumes no clause of Φ, since ψ subsumed none.

This completes the proof that (i) and (ii) deliver *all* the prime implicants. The proof of the converse, namely that (i) and (ii) when continued as long as possible yield an alternation of prime implicants *only*, can now be added in a few words. Since $\alpha\phi \vee \bar{\alpha}\psi$ is equivalent to $\alpha\phi \vee \bar{\alpha}\psi \vee \phi\psi$, clearly (ii) is, like (i), an equivalence transformation. Accordingly the alternation obtained from a formula Φ by applying (i) and (ii) as long as possible can be depended upon to be an alternation of clauses each of which implies Φ. But, as we just finished proving in the preceding paragraph, every prime implicant is a clause. Accordingly any clause that is not a prime implicant subsumes another clause which is a prime implicant. But this cannot happen; (i) would be applicable again.

I shall discuss, for the remainder of the paper, the business of moving from the alternation of all prime implicants to a shortest equivalent (as always, in alternational normal form). This is wholly a matter of dropping dispensable clauses. There is, moreover, this *test of dispensability* of a single clause ϕ: see whether ϕ implies the remainder, Ψ, of the alternation. This may be quickly decided by testing Ψ for truth when the letters affirmed in ϕ are marked true and those negated in ϕ are marked false.

The dropping of one dispensable clause, however, can render another originally dispensable clause indispensable to the remaining alternation. We want rather to find the largest simultaneously dispensable combination of clauses. Certain aids to this end will now be noted.

We can get a head start by reviewing the applications of (ii) which were made in arriving at the alternation of all prime implicants. We bracket out, in a body, all clauses that were added by (ii) after the last application of (i). In our example we get:

$$(5) \qquad ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee qr\bar{s} \vee pqr \vee [r\bar{s}t \vee prt].$$

These brackets serve as a reminder that the clauses enclosed are dispensable, and not only singly but jointly; for, the alternation which includes those bracketed clauses was originally got, from itself minus those clauses, by an equivalence transformation.

We must not, however, without further check, bracket a clause ϕ ('pqr', in the example) which was added by (ii) prior to a use of (i); for the subsequent use of (i) may, by banishing another clause, have rendered ϕ indispensable.

The next move, rather, is to subject each unbracketed clause to the test of dispensability formulated above, and bracket individually each clause which meets the test. The result, in the case of (5), is:

(6) $$ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee [qr\bar{s}] \vee [pqr] \vee [r\bar{s}t \vee prt].$$

This done, we can conclude a good deal about any shortest equivalent. It is bound to retain the unbracketed part. (This is what was called the *core* in PSTF, p. 527.) Its remaining clauses, if any, will be certain of those shown in brackets; and not all of them, since any one bracketed portion, at least, is dispensable. If, as commonly happens, there are no bracketings or just one, then the shortest equivalent is the core itself. It is only in cases such as (6), with an unusual lot of bracketing, that any serious exhaustion of possibilities remains to be done. An approach to such cases (doubtless susceptible, however, to streamlining) is the following.

A bracketed clause ϕ may or may not imply the core; all we know is that ϕ implies the whole alternation minus ϕ itself. If ϕ does imply the core, then ϕ should be cancelled for good. Any shortest equivalent, retaining the whole core as it does, is bound to omit any clause that implies the core. Such clauses are *absolutely* dispensable—independently of the omission or retention of other bracketed clauses.

So each bracketed clause should be tested individually (in the quick way lately mentioned) to see if it implies the core; and each which does should be deleted. In our example (6), none of the four bracketed clauses proves to imply the core. But here is an example where the phenomenon does occur:

$$pq \vee pr \vee \bar{p}s \vee \bar{r}t \vee [pt] \vee [qs].$$

Here, in fact, each of the bracketed prime implicants proves absolutely dispensable; only the core remains.

In general the remaining task, if any, in finding the shortest equivalents of a formula, is to test combinations of bracketed passages for joint dispensability. This can be done as follows. To see whether $\phi_1, \cdots, \phi_n$ are jointly dispensable in $\Psi \vee \phi_1 \vee \cdots \vee \phi_n$, hence whether $\phi_1 \vee \cdots \vee \phi_n$ implies Ψ, check separately for each i (by the quick method noted earlier) to see whether ϕ_i implies Ψ. If ϕ_i implies Ψ for each i, then and only then $\phi_1 \vee \cdots \vee \phi_n$ implies Ψ and is dispensable *in toto*.

There is evident strategy in testing big combinations ahead of smaller ones. Only partial combinations want testing, however; the preceding search for absolutely dispensable clauses was *ipso facto* a test of whether the *whole* bracketed portion was dispensable.

In the case of (6), what we find is that '$qr\bar{s} \vee r\bar{s}t \vee prt$' implies the alternation of the remaining four clauses, and similarly for '$pqr \vee r\bar{s}t \vee prt$'; so we end up with two shortest equivalents:

$$ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee pqr, \qquad ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee qr\bar{s}.$$

Despite the evident advantage of our new method over the method of developed normal forms and tables in PSTF, we should continue to exploit the separation expedient noted on page 529 of PSTF when we can.

ON SPHERICAL DRAWING AND COMPUTATION

MILTON FELSTEIN, U.S. Naval Gun Factory, Washington, D.C.

1. Introduction. Pictorial spherical diagrams (Figure 1) are tedious to draw by conventional methods and, as a result, are frequently found approximated or sketched, even in textbooks and treatises. Such diagrams will be recognized as axonometric projections of a sphere intersected by planes passing through its center. Besides their pictorial value, such diagrams provide a means for the geometrical solution of spherical triangles.

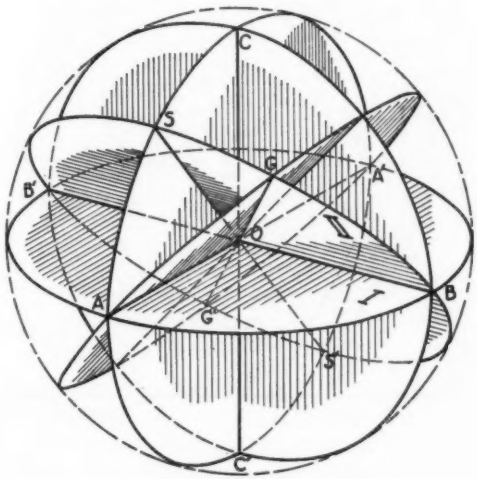

FIG. 1

The planes shown in Figure 1 intersect the sphere in great circles, and these appear on the drawing as *derived ellipses* of the respective great circles *primitive* to them. Orthogonal diameters of the great circles appear as *conjugate diameters* of the derived ellipses, for example, AA', CC'. The dihedral angle between two planes (I and II) having a given line of intersection (BB') is measured by the intercepted arc (AS) of the great circle ($ACA'C'$) normal to BB'.

Two characteristic problems are involved in drawing diagrams like Figure 1:

(1) Drawing an ellipse, given a pair of diameters. These are usually conjugate diameters, like AA', CC', but may be non-conjugate, like AA', GG'. The required ellipse will represent the great-circle intersection of the sphere with the plane determined by the given diameters. (2) Laying off on an ellipse an arc having a given angular measure on the primitive great circle. One important application of this problem is to normal great circles like $ACA'C'$, whereby a plane (II) making the given angle (AS) with a second plane (I) is determined.

Simple and rapid methods are described here for solving these problems using a grid of spherical coordinates slipped under the tracing paper (Figure 2). Conventional methods for solving problem 1 can do so only when the given diameters are conjugate, but the grid method can handle the general case.

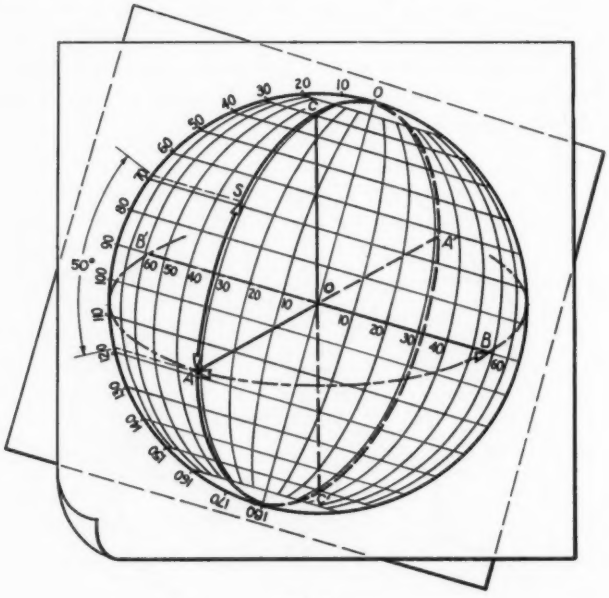

FIG. 2

2. Starting the drawing. The drawing is usually started with the projection of a unit sphere containing three orthogonal diameters (AA', BB', CC') representing three unit coordinate axes (Figure 3). The trimetric system shown is a convenient one for general use. Dotted lines show radii behind the plane of the circle representing the sphere outline (the so-called ***meridian*** plane). In the ***isometric*** system the axes are equally spaced (120°) about the center, with length of each equal to 0.815 of the diameter. Methods for computing such systems are properly part of the doctrine of axonometric scaling, and are discussed later. It is noteworthy that the grid shown in Figure 2 provides the simplest method

of computing axonometric scales.

3. Characteristic problem 1. To show the great-circle intersections of planes determined by the coordinate axes with the sphere brings up the first characteristic problem in the common form: Given the conjugate diameters AA', CC', draw the ellipse. To do this, the sphere is centered on a grid of the same size, and rotated until some one ellipse of the grid, interpolated if necessary, passes through all of the diametral end-points, A, C, A', C'. There are actually two ellipses that will do this on the drawing sheet, but one of them will be absurd so far as passing through the given points in space. (Monitor by considering visibility relative to the meridian plane.)

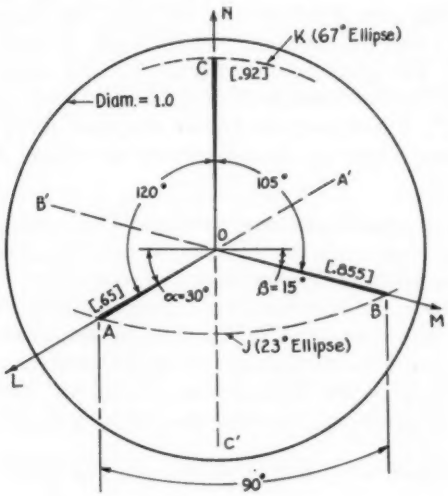

FIG. 3

Conventional methods for solving this problem will be found in books on drafting (see Bibliography at end of article). Two of these methods, the so-called "parallelogram" and "circle" methods, involve point-by-point plotting of the required ellipse. In a third, the "trammel" method, a triangle is constructed which is moved in a certain manner to produce the required ellipse as a locus.

The grid method will work equally well for any pair of diameters, whether or not conjugate. In Figure 1 is shown the ellipse $AGA'G'$ established from the non-conjugate diameters AA', GG'.

For later use, notice that the minor axis of the ellipse $ACA'C'$, lying along the equatorial line of the grid, is in the direction of the coordinate axis BB', thus representing perpendicularity in space relative to the primitive of the ellipse. (Do not confuse axes of the ellipse with coordinate axes of the diagram or with conjugate diameters.)

4. Characteristic problem 2. Given the ellipse $ACA'C'$, to lay off an arc AS representing 50° on the primitive great circle. When chords are drawn in the direction of the minor axis of an ellipse representing a great circle of a sphere, they cut arcs of equal angular measure from the ellipse and the sphere-outline circle. Accordingly, the sphere is centered on the grid and rotated until the ellipse $ACA'C'$ coincides with some one ellipse of the grid, interpolated if necessary. Using the parallels of the grid as projectors, the required arc on the ellipse is then established from the graduated outline.

The arrangement in Figure 2 illustrates, in passing, a second and faster method of laying out the particular value 90° if there is represented on the drawing a diameter perpendicular in space to the primitive of the given ellipse. The given ellipse is in this case $ABA'B'$ (not $ACA'C'$), the diameter through the initial point is AA', and the diameter perpendicular in space to the primitive of $ABA'B'$ is CC'. The grid and drawing are rotated until some one ellipse ($ACA'C'$) of the grid passes through all of the end points of the two orthogonal diameters AA', CC'. The equatorial line of the grid, being perpendicular in space to each of these diameters, then intersects the ellipse $ABA'B'$ at the required point B.

5. Computation of sphere-and-axes systems. The fundamental problem of axonometric scaling is to compute the relative projected lengths of unit axes orthogonal in space, given their directions on the drawing. The result is a set of *axonometric scales*. In drawing spherical diagrams, the relative true length of the unit is also needed, to establish the diameter of the sphere.

There is to be found in the literature a geometrical construction yielding true and projected lengths (see Bibliography), which will not be repeated here. Another method is to compute them from the following equations:

$$AA' : BB' : CC' : \text{Diam.} = \sqrt{\sin 2\beta} : \sqrt{\sin 2\alpha} : \sqrt{\sin (2\alpha + 2\beta)}$$
$$: \sqrt{\tfrac{1}{2}[\sin 2\beta + \sin 2\alpha + \sin (2\alpha + 2\beta)]}.$$

If one starts with a set of axonometric scales, such as are found ready-made in drafting books, the relative true length of the unit is thus given by the square root of half the sum of squares of the given scale values.

The simplest method of computing sphere-and-axes systems, or axonometric scales alone, is with the grid, as shown in Figure 3. The grid is centered under the intersection (O) of the three given direction lines (OL, OM, ON) with its parallels vertical. An ellipse is located whose intersections (A, B) with the two non-vertical direction lines (OL, OM) are 90° apart as indicated by the parallels. These intersections establish the required unit lengths in these directions. The point establishing the unit length in the vertical direction is the intersection (C) of the vertical direction line (ON) with the ellipse (K) whose degree reading is the complement of that of the 23°-ellipse (J), that is, the 67°-ellipse.

6. Geometrical solution of spherical triangles. If the triangle is placed as shown in Figure 4, and only elements x, y, z, X, and Z are used, the parts can be measured or laid off directly using the grid. Element y is measured from the outline circle, elements z and X when the grid has its common major axis along UV, and elements x and Z when it is along $U'V'$. To avoid use of element Y may require working the problem a second time after reassignment of knowns and unknowns. The additional construction required to measure Y directly is shown in the figure, where DUP and $DU'T$ are 90°-arcs, and the ellipse $FPTE$ has its minor axis in the direction OD. The arc TP provides the required angular measure of Y.

When solving a triangle given the three angles, the three sides of the corresponding polar triangle are calculated as the respective supplements of the angles of the given triangle. The polar triangle is solved for its angles, using the method given, and the supplements of these are then calculated for the required sides of the given triangle.

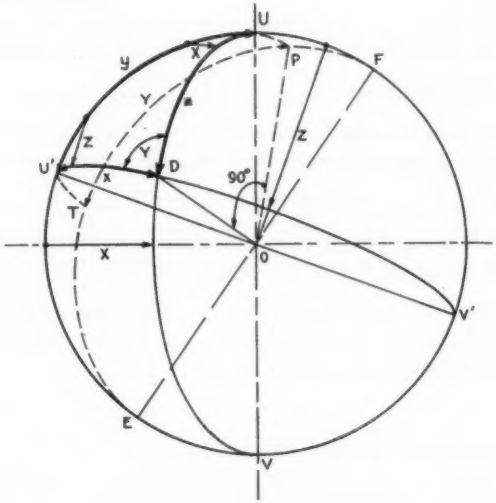

FIG. 4

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ON THE POSTULATES DEFINING A GROUP

J. V. WHITTAKER, University of California at Los Angeles

1. Introduction. Although a group is usually defined in terms of the operation $x+y$, when written additively, there are some concepts connected with a group, such as congruence and subgroup, which are more readily definable in terms of the inverse operation $x-y=x+(-y)$. It is possible to formulate the postulates for a group in terms of the latter operation and then to define the former operation in terms of the latter. The object of this paper is to carry out that formulation and to consider some of its consequences.

2. The postulates. Let G be a set of elements over which is defined a binary operation written $x-y$ and satisfying, for any elements x, y, and z of G, the following postulates:

I. $x-y \in G$,

II. *There is an element* 0 *of* G *such that* $x-y=0$ *if, and only if,* $x=y$,

III. $(x-z)-(y-z)=x-y$.

This operation will be called *subtraction* and $x-y$ the *difference* of x and y. A second operation, called *addition* and written $x+y$, will be defined as

$$x + y = x - (0 - y). \tag{1}$$

THEOREM 1. *If* G *satisfies* I, II, *and* III, *and if addition is defined by* (1), *then* G *is a group. Conversely, if* G *is a group, then* I, II, *and* III *are satisfied.*

Proof. It is evident that if, in a group, $x-y$ is defined to be $x+(-y)$, then I, II, and III are satisfied. Conversely, G is obviously closed under the operation of addition defined by (1). Setting $z=x$ in III,

$$0 - (y - x) = x - y. \tag{2}$$

From III can be derived the related identities

$$(x + z) - (y + z) = x - y, \tag{3}$$

$$(x - z) + (z - y) = x - y. \tag{4}$$

A cancellation law also follows from II and III:

$$x - z = y - z \text{ if, and only if, } x = y. \tag{5}$$

Setting $y=0$, $z=0$ in III,

$$(x - 0) - 0 = x - 0;$$

then from (5),

$$x - 0 = x.$$

Setting $y=0$ in (2),

$$0 - (0 - x) = x.$$

Now the element 0 is a left identity for x, and $0-x$ a left inverse of x, for

$$0 + x = 0 - (0 - x) = x,$$
$$(0 - x) + x = (0 - x) - (0 - x) = 0.$$

Setting $y=0$, $z=y$ in (3),

$$(x + y) - y = x.$$

The associative law now follows:

$$\begin{aligned} x + (y + z) &= ((x + y) - y) + (y - (0 - z)) \\ &= (x + y) - (0 - z) = (x + y) + z. \end{aligned}$$

This completes the proof.

In considering the logical independence of I, II, and III, we shall show that I, III, and the statements

IIa. *There is an element* 0 *such that if* $x-y=0$, *then* $x=y$,

IIb. *There is an element* 0 *such that if* $x=y$, *then* $x-y=0$

are logically independent. In fact, table A fails to satisfy I, table B with $a=0$ does not fulfill IIa, with $b=0$ it does not fulfill IIb, and table C fails to satisfy III, while in each case the three remaining postulates hold. The independence of IIa and IIb is to be interpreted in the sense that G may have an element 0 satisfying one of these postulates but not the other.

$-$	a	b	c
a	a	c	b
b	b	a	c
c	c	b	a

Table A.

$-$	a	b
a	a	a
b	a	a

Table B.

$-$	a	b	c
a	a	b	c
b	b	a	c
c	c	b	a

Table C.

3. Postulates for a commutative group. The commutative law may be written in the form

IV. $x-(x-y)=y$,

for IV is obviously equivalent to $x+y=y+x$ in any group.

THEOREM 2. *A necessary and sufficient condition for* G *to be a commutative group is that it satisfy* I, III, *and* IV.

Proof. It only remains to show that I, III, and IV imply II. We first observe that

$$(x - (x - y)) - (x - (x - y)) = x - x = y - y;$$

this element we shall call 0. Setting $y=x$ in IV,

$$x - 0 = x.$$

Now if $x-y=0$, then

$$x - (x - y) = x = y.$$

Tables A and C show that I, III, and IV are also logically independent.

The condition

V. $(x-z)-(x-y)=y-z$

resembles III in form but is really much stronger, as the following theorem shows.

THEOREM 3. *A necessary and sufficient condition for G to be a commutative group is that it satisfy* I, II, *and* V.

Proof. The condition is evidently necessary. Conversely, setting $x=0$ in V, we conclude that

$$0 - y = 0 - z \quad \text{if, and only if,} \quad y = z. \tag{6}$$

Setting $y=0$, $z=x$ in V,

$$0 - (x - 0) = 0 - x.$$

From (6) it follows that

$$x - 0 = x.$$

Setting $z=0$ in V,

$$x - (x - y) = y. \tag{7}$$

Finally, replacing x by $x-z$, y by $x-y$ in (7),

$$(x - z) - (y - z) = x - y.$$

This proves the sufficiency of the condition. Tables A, B, and C suffice to prove the logical independence of I, II, and V.

4. Finite group tables. Let G be a finite set of elements $a_1, \cdots, a_n$ over which is defined a subtraction, and let the differences a_i-a_j be arranged in a table in which

$$a_i - a_j = a_{ij}, \qquad (i, j = 1, \cdots, n).$$

It will be convenient, in deciding whether or not the table represents a group, to see if it satisfies I, II, and III. Postulates I and II can be checked by inspection: the elements on the main diagonal must be the same and must appear no-

where else in the table. In order for III to be fulfilled, the differences of corresponding elements in any two given rows must equal the difference of the row headings, that is,

$$a_{ik} - a_{jk} = a_i - a_j, \qquad (i, j, k = 1, \cdots, n).$$

If the table is to represent a commutative group, then it must satisfy V in addition to I and II, that is, the differences of corresponding elements in any two given columns must equal the difference of the column headings reversed:

$$a_{ij} - a_{ik} = a_k - a_j, \qquad (i, j, k = 1, \cdots, n).$$

Checking III or V is somewhat simpler than checking the associative law in the addition table.

The postulate

IIc. *There is an element* 0 *of* G *such that* $0-0=0$ *and such that if* $x-y=0$, *then* $x=y$,

although weaker than II in general, is equally strong in the case of finite groups, as the following theorem demonstrates.

THEOREM 4. *A necessary and sufficient condition for a finite set* G *to be a group is that it satisfy* I, IIc, *and* III.

Proof. Let the elements of G be $a_1, \cdots, a_n$. It remains only to show that $a_{ii}=0$ for $1 \leqq i \leqq n$. Let the elements of G be so ordered that $a_{ii}=0$ implies $1 \leqq i \leqq m$, and suppose that $m<n$. If $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$, then

$$a_{ij} - a_{ij} = a_i - a_i = 0,$$

so that $a_{ij}=a_k$, where $1 \leqq k \leqq m$. Since 0 is not among the elements $a_{i,m+1}$ for $1 \leqq i \leqq m$, two of them must be equal, say $a_{r,m+1}=a_{s,m+1}$. But then

$$a_{r,m+1} - a_{s,m+1} = a_r - a_s = 0,$$

or $a_r=a_s$, which is a contradiction. Hence, $m=n$.

5. Refinement of the postulates. Let G be a set over which are defined an equivalence relation, called congruence and written $x \equiv y$, and an operation, written $x-y$, which satisfy I, II, III, with equality replaced by congruence in II and III, and the postulate

VI. *If* $v \equiv x$ *and* $w \equiv y$, *then* $v-w \equiv x-y$.

Then the equivalence classes of G form a group, by Theorem 1. In this sense, we shall say that G itself is a group.

Postulate VI can be written as the conjunction of the statements

VIa. *If* $v \equiv x$, *then* $v-w \equiv x-w$,

VIb. *If* $w \equiv y$, *then* $x-w \equiv x-y$,

whence it follows that II and III imply VIa, and II and V, with equality replaced by congruence in the latter, imply VIa and VIb. Thus, in stating the postulates for a group, VIa can be dispensed with in the general case and VI in the commutative case.

Of the postulates I, II, III, and VIb, III embodies more of the group character of G than the rest, for it asserts the invariance of the difference $x-y$ as x and y vary by the same amount. This property can be stated in the stronger form

VII. *$v-w\equiv x-y$ if, and only if, there is a z such that $v\equiv x-z$ and $w\equiv y-z$.*

THEOREM 5. *A necessary and sufficient condition for G to be a group is that it satisfy* I *and* VII.

Proof. The necessity of the condition follows at once. In particular, if $v-w\equiv x-y$, then z can be taken to be $-w+y$. Suppose, conversely, that I and VII hold. For any three elements x, y, and z, let $v=x-z$, $w=y-z$. Then

$$v \equiv x - z, \qquad w \equiv y - z,$$

$$v - w = (x - z) - (y - z) \equiv x - y,$$

and III is established. Since $x-y\equiv x-y$, there is a z such that $x\equiv x-z$, $y\equiv y-z$. Now if $w\equiv y$, then $x\equiv x-z$, $w\equiv y-z$, and $x-w\equiv x-y$. Thus VIb is established. For each x, let $E(x)$ be the set of elements z such that $x\equiv x-z$. Evidently $E(x)$ is not empty. If $z, z'\in E(x)$, then $x\equiv x-z'$, $z-z'\equiv z-z'$, and

$$x - (z - z') \equiv x - z \equiv x,$$

hence, $z-z'\in E(x)$. Since $x-z\equiv x-z'$, there is a z'' such that $x\equiv x-z''$, $z\equiv z'-z''$. Hence,

$$z - z \equiv z' - z'.$$

Now choose $x_0\in G$ and $z_0\in E(x_0)$, and set $z_0-z_0=0$. For each x, there is a z such that

$$z \in E(x) \cap E(x_0),$$

so that

$$z - z \in E(x) \cap E(x_0).$$

Therefore,

$$x \equiv x - (z - z) \equiv x - 0.$$

If $x-y\equiv 0$, then $x-y\equiv 0-0$, and there is a z such that $x\equiv 0-z$, $y\equiv 0-z$; hence, $x\equiv y$. Conversely, if $x\equiv y$, then $x-y\equiv x-x$. Since $(x-x)-(x-x)\equiv x-x$, it follows that $x-x\in E(x-x)$ and that $(x-x)-(x-x)\equiv 0$. Therefore, $x-y\equiv 0$, and II is fulfilled. We have thus established the sufficiency of the condition, and the proof is complete.

LINEAR DIFFERENCE EQUATIONS AND THE DIRICHLET SERIES TRANSFORM

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The solution of linear differential equations by means of the Laplace transform is now commonplace. Linear difference equations with constant coefficients also can usually be solved by means of the Laplace transform. The Dirichlet series transform, however, is usually simpler and seems a much more natural instrument for the solution of difference equations. Apparently it has been overlooked.

We let t be an integer and then let

$$f(s) = D\{a(t)\} = \sum_{t=0}^{\infty} m^{-st}a(t), \qquad m > 1, s > 1. \tag{1}$$

This is the basic transformation of the paper. We assume the series convergent. We note in passing that the transformation is linear.

Let

$$\Delta F(t) = F(t+1) - F(t). \tag{2}$$

We call attention to the summation by parts formula

$$\sum_{t=a}^{b} u(t)\Delta v(t) = u(t)v(t)\Big]_a^{b+1} - \sum_{t=a}^{b} v(t+1)\Delta u(t). \tag{3}$$

We apply this formula to Dirichlet series (1). Let $u(t) = a(t)$,

$$\Delta v(t) = m^{-st}, \qquad v(t) = \frac{m^{-st}}{m^{-s} - 1}.$$

We get equation (4) below

$$f(s) = D\{a(t)\} = a(t)\frac{m^{-st}}{m^{-s} - 1}\Big]_0^{\infty} - \sum_{t=0}^{\infty} \frac{m^{-s(t+1)}}{m^{-s} - 1}\Delta a(t). \tag{4}$$

$$= -a(0)\frac{m^{s}}{1 - m^{s}} - \frac{1}{1 - m^{s}} \sum_{t=0}^{\infty} m^{-st}\Delta a(t).$$

We now repeat the operation of summation by parts on the final sum in formula (4). We notice that this sum is the same as that in (1) with $a(t)$ replaced by $\Delta a(t)$. We get the following result:

$$f(s) = -a(0)\frac{m^{s}}{1 - m^{s}} + [\Delta a(0)]\frac{m^{s}}{(1 - m^{s})^2} + \frac{1}{(1 - m^{s})^2} \sum_{t=0}^{\infty} m^{-st}\Delta^2 a(t). \tag{5}$$

The operation can be repeated indefinitely, and we get the following formula:

$$
\begin{aligned}
f(s) = &-\frac{m^s}{1-m^s}a(0)+\frac{m^s}{(1-m^s)^2}\Delta a(0)-\frac{m^s}{(1-m^s)^3}\Delta^2 a(0)\\
&+\cdots+(-1)^n\frac{m^s}{(1-m^s)^n}\Delta^{n-1}a(0)+(-1)^n\frac{1}{(1-m^s)^n}\sum_{t=0}^{\infty}m^{-st}\Delta^n a(t).
\end{aligned}
\tag{6}
$$

The next step is to solve for the respective transforms:

$$D\{a(t)\}=f(s) \tag{7}$$

$$D\{\Delta a(t)\}=-m^s a(0)-(1-m^s)f(s) \tag{8}$$

$$D\{\Delta^2 a(t)\}=-m^s\Delta a(0)+m^s(1-m^s)a(0)+(1-m^s)^2 f(s) \tag{9}$$

. . . . . .

$$
\begin{aligned}
D\{\Delta^n a(t)\}=&-m^s\Delta^{n-1}a(0)+m^s(1-m^s)\Delta^{n-2}a(0)\\
&+\cdots+(-1)^{n-2}m^s(1-m^s)^{n-3}\Delta^2 a(0)+(-1)^{n-1}m^s(1-m^s)^{n-2}\Delta a(0)\\
&+(-1)^n m^s(1-m^s)^{n-1}a(0)+(-1)^n(1-m^s)^n f(s).
\end{aligned}
\tag{10}
$$

Even easier are the transforms of successive values. Let j be a positive integer;

$$
\begin{aligned}
D\{a(t+j)\} &= m^{js}\sum_{t=0}^{\infty}m^{-s(t+j)}a(t+j)\\
&= m^{js}\sum_{t=0}^{\infty}m^{-st}a(t)-m^{js}\sum_{t=0}^{j-1}m^{-st}a(t)\\
&= m^{js}D\{a(t)\}-m^{js}\sum_{t=0}^{j-1}m^{-st}a(t).
\end{aligned}
\tag{11}
$$

We now prepare a table of transforms for the most frequently occurring functions.

I. $a(t)=1$

$$D\{a(t)\}=\sum_{t=0}^{\infty}m^{-st}=\frac{m^s}{m^s-1}.$$

II. $a(t)=t$

$$
\begin{aligned}
D\{a(t)\} &= \sum_{t=0}^{\infty}m^{-st}t=t\frac{m^{-st}}{m^{-s}-1}\Bigg]_0^{\infty}-\sum_{t=0}^{\infty}\frac{m^{-st}(t+1)}{m^{-s}-1}\\
&= -\frac{m^{-s}}{m^{-s}-1}\sum_{t=0}^{\infty}m^{-st}=\frac{1}{m^s-1}D\{1\}=\frac{m^{-s}}{(m^s-1)^2}.
\end{aligned}
$$

III. $a(t)=t(t-1)=t^{(2)}$

$$D\{a(t)\} = \sum_{t=0}^{\infty} m^{-st}t^{(2)} = -\frac{1}{m^{-s}-1}\sum_{t=0}^{\infty} m^{-s(t+1)}2t$$

$$= -\frac{2m^{-s}}{m^{-s}-1}\sum_{t=0}^{\infty} m^{-st}t = \frac{2}{m^s-1}D\{t\} = \frac{2m^s}{(m^s-1)^3}.$$

. . . . . .

IV. $a(t)=t(t-1)\cdots(t-r+1)=t^{(r)}$

By the use of formula (3) as under III,

$$D\{a(t)\} = \frac{r}{m^s-1}D\{t^{(r-1)}\} = \frac{r!m^s}{(m^s-1)^{r+1}}.$$

V. $a(t)=t^r$ where r is a positive integer

Write t^r as a linear function of $t^{(r)}, t^{(r-1)}, \cdots, 1$ and use IV.

VI. $a(t)=k^t$ where $|k|<m^s$

$$D\{a(t)\} = \sum_{t=0}^{\infty}(m^{-s}k)^t = \frac{m^s}{m^s-k}.$$

VII. $a(t)=t^{(r)}k^t$

$$D\{a(t)\} = \sum_{t=0}^{\infty}(m^{-s}k)^t\cdot t^{(r)}.$$

Replace m^{-s} in IV by $m^{-s}k$. We get

$$D\{a(t)\} = \frac{r!m^s k^r}{(m^s-k)^{r+1}}.$$

VIII. $a(t)=\sin qt$

$$D\{a(t)\} = \frac{1}{2i}\sum_{t=0}^{\infty} m^{-st}e^{qit} - \frac{1}{2i}\sum_{t=0}^{\infty} m^{-st}e^{-qit}$$

$$= \frac{1}{2i}\sum_{t=0}^{\infty}(m^{-s}e^{qi})^t - \frac{1}{2i}\sum_{t=0}^{\infty}(m^{-s}e^{-qi})^t$$

$$= -\frac{1}{2i}\left[\frac{1}{m^{-s}e^{qi}-1} - \frac{1}{m^{-s}e^{qi}-1}\right] = \frac{m^s\sin q}{1+m^{2s}-2m^s\cos q}.$$

IX. $a(t)=\cos qt$

$$D\{a(t)\} = -\frac{1}{2}\left[\frac{1}{m^{-s}e^{qi}-1} + \frac{1}{m^{-s}e^{-qi}-1}\right] = \frac{m^{2s}-m^s\cos q}{1+m^{2s}-2m^s\cos q}.$$

X. $a(t)=t^{(r)}\sin qt$

This transformation is worked out by means of exponentials and VII. If

$r>1$ it is advised to replace $\sin qt$ by exponentials in the beginning and to carry the work through to the desired solution. A return can then be made to trigonometric functions if desired.

However, let

$$(m^s - \cos q + i \sin q)^{r+1} = X + Yi.$$

We then find

$$D\{a(t)\} = r!m^s \frac{Y \cos rq + X \sin rq}{(m^{2s} + 1 - 2m^s \cos q)^{r+1}}.$$

In particular,

$$D\{t \sin qt\} = \frac{m^{3s} \sin q - m^s \sin q}{(m^{2s} + 1 - 2m^s \cos q)^2}.$$

XI. $a(t) = t^{(r)} \cos qt$

$$D\{a(t)\} = r!m^s \frac{X \cos rq - Y \sin rq}{(m^{2s} + 1 - 2m^s \cos q)^{r+1}}.$$

In particular,

$$D\{t \cos qt\} = \frac{(m^{3s} + m^s) \cos q - 2m^{2s}}{(m^{2s} + 1 - 2m^s \cos q)^2}.$$

XII. $a(t) = k^t \sin qt$, $|k| < m^s$

$$D\{a(t)\} = \sum_{t=0}^{\infty} (m^{-s}k)^t \sin qt = \frac{m^s k^{-1} \sin q}{1 + m^{2s}k^{-2} - 2m^s k^{-1} \cos q}.$$

XIII. $a(t) = k^t \cos qt$, $|k| < m^s$

$$D\{a(t)\} = \frac{m^{2s}k^{-2} - m^s k^{-1} \cos q}{1 + m^{2s}k^{-2} - 2m^s k^{-1} \cos q}.$$

XIV. $a(t) = t^{(r)}k^t \sin qt$, $|k| < m^s$

Replace m^s in X by $m^s k^{-1}$.

In particular,

$$D\{tk^t \sin qt\} = \frac{m^{3s}k^{-3} \sin q - m^s k^{-1} \sin q}{(m^{2s}k^{-2} - 2m^s k^{-1} \cos q + 1)^2}.$$

XV. $a(t) = t^{(r)}k^t \cos qt$, $|k| < m^s$

Replace m^s in XI by $m^s k^{-1}$. In particular,

$$D\{tk^t \cos qt\} = \frac{(m^{3s}k^{-3} + m^s k^{-1}) \cos q - 2m^{2s}k^{-2}}{(m^{2s}k^{-2} - 2m^s k^{-1} \cos q + 1)^2}.$$

Example.

$$\Delta y(t) - y(t) = t.$$

Apply the Dirichlet series transformation to both sides of this equation and we get the following equation:

$$-y(0)m^s + (m^s - 1)f(s) - f(s) = \frac{m^s}{(m^s - 1)^2}.$$

Solve this for $f(s)$. We get

$$f(s) = \frac{m^s}{(m^s - 1)^2(m^s - 2)} + y(0)\frac{m^s}{m^s - 2}$$

$$= -\frac{m^s}{m^s - 1} - \frac{m^s}{(m^s - 1)^2} + \frac{m^s}{m^s - 2} + y(0)\frac{m^s}{m^s - 2}.$$

We now refer to our tabulated transforms and obtain the solution of our equation:

$$y(t) = -1 - t + 2^t + y(0)2^t.$$

MATHEMATICAL NOTES

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Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

NOTE ON A CONVERGENCE PROBLEM

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In [1], A. Shields suggested the following as a Research Problem:

If $a_1 < a_2 < \cdots$ are positive integers, if C is compact, and if $\sin a_n x \to 0$ for all x in C, prove that the convergence must actually be uniform.

A simple counterexample to this assertion is afforded by taking $a_n = n!$ $(n = 1, 2, 3, \cdots)$ and C to be the set of points $0, \pi, \pi/2, \pi/3, \pi/4, \cdots$. Since $n!/m$ is an integer for $n > m$, $\sin a_n x = 0$ for all $n > N_x$ and $\sin a_n x \to 0$ for all x in C. Yet for any fixed n there is an x in C for which $\sin a_n x = 1$, namely $x = \pi/(2n!)$. Convergence is therefore not uniform.

Reference

1. Bull. Amer. Math. Soc., vol. 60, 1954, p. 589.

A NOTE ON HERMITE POLYNOMIALS

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Feldheim [2] has proved the formula

$$H_m(x)H_n(x) = \sum_{r=0}^{\min(m,n)} 2^r r!\binom{m}{r}\binom{n}{r} H_{m+n-2r}(x), \tag{1}$$

where $H_m(x)$ is the Hermite polynomial defined by

$$H_m(x) = \sum_{2s\leq m} (-1)^s (2x)^{m-2s} \frac{m!}{s!(m-2s)!}. \tag{2}$$

For other proofs of (1) see Watson [3] and Burchnall [1]. We wish to point out that (1) can be proved rapidly by induction. Indeed if $U_\mu(x)$ denotes an arbitrary solution of the equation

$$U_{\mu+1} = 2xU_\mu - 2\mu U_{\mu-1}, \tag{3}$$

where μ is an arbitrary complex number, then we shall prove

$$H_m(x)U_\mu(x) = \sum_{r=0}^{m} 2^r r!\binom{m}{r}\binom{\mu}{r} U_{m+\mu-2r}(x). \tag{4}$$

Since $H_{m+1}(x) = 2xH_m(x) - 2mH_{m-1}(x)$ it is clear that (4) includes (1).

Since the parabolic cylinder function $D_\mu(x)$ [4; Chapter 16] satisfies $D_{\mu+1}(x) = xD_\mu(x) - \mu D_{\mu-1}(x)$, it follows that $U_\mu = 2^{\mu/2} D_\mu(2^{1/2}x)$ satisfies (3).

We now prove (4). For $m=0$, (4) is obvious. For $m=1$, (4) reduces to (3). We accordingly assume the truth of (4) for the value m; then

$$\begin{aligned}
H_{m+1}(x)U_\mu(x) &= \{2xH_m(x) - 2mH_{m-1}(x)\}U_\mu(x) \\
&= 2x\sum_{r=0}^{m} 2^r r!\binom{m}{r}\binom{\mu}{r} U_{m+\mu-2r}(x) - 2m\sum_{r=0}^{m-1} 2^r r!\binom{m-1}{r}\binom{\mu}{r} U_{m+\mu-1-2r}(x) \\
&= \sum_{r=0}^{m} 2^r r!\binom{m}{r}\binom{\mu}{r}\{2xU_{m+\mu-2r}(x) - 2(m+\mu-2r)U_{m+\mu-1-2r}(x)\} \\
&\quad + 2\sum_{r=0}^{m} 2^r r!\binom{m}{r}\binom{\mu}{r}(\mu - r)U_{m+\mu-1-2r}(x) \\
&= \sum_{r=1}^{m} 2^r r!\binom{m}{r}\binom{\mu}{r} U_{m+\mu+1-2r}(x) + \sum_{r=1}^{m+1} 2^r r!\binom{m}{r-1}\binom{\mu}{r} U_{m+\mu+1-2r}(x) \\
&= \sum_{r=0}^{m+1} 2^r r!\binom{m+1}{r}\binom{\mu}{r} U_{m+\mu+1-2r}(x).
\end{aligned}$$

This evidently completes the induction.

In a similar way we can prove the companion formula

$$U_{m+\mu}(x) = \sum_{r=0}^{m} (-1)^r 2^r r! \binom{m}{r}\binom{\mu}{r} H_{m-r}(x) U_{\mu-r}(x). \tag{5}$$

It follows from (4) that

$$\frac{H_m(x)U_{\mu+1}(x) - H_{m+1}(x)U_\mu(x)}{2(m-\mu)} = \sum_{r=0}^{m} 2^r r! \binom{m}{r}\binom{\mu}{r} U_{m+\mu-1-2r}(x), \tag{6}$$

$$\frac{(m-\mu+1)H_m(x)U_{\mu+2}(x) - 2(m-\mu)H_{m+1}(x)U_{\mu+1}(x) + (m-\mu-1)H_{m+2}(x)U_\mu(x)}{4(m-\mu+1)(m-\mu)(m-\mu-1)}$$

$$= \sum_{r=0}^{m} 2^r r! \binom{m}{r}\binom{\mu}{r} U_{m+\mu-2-2r}(x),$$

and so on.

Added in proof. The formulas (1), (4) and (5) occur in slightly different notation in a paper by N. Nielsen, *Recherches sur les polynomes d'Hermite*, Det kgl. Danske Videnskabernes Selskab, Mathematisk-fysiske Meddelelser, I, 6, pp. 1–78, in particular pp. 31–33.

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3. G. N. Watson, A note on the polynomials of Hermite and Laguerre, J. London Math. Soc. vol. 13, 1938, pp. 29–32.

4. E. T. Whittaker and G. N. Watson, Modern Analysis, 4th Edition, Cambridge, 1927.

NOTES ON MATRIX THEORY—VII

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1. Introduction. There is a meta-mathematical theorem to the effect that to every multiplicative theorem in analysis, there corresponds an additive one obtained by a limiting process. The purpose of this note is to show how this principle may be used to derive one inequality of Ky Fan's from another inequality of multiplicative type.

Let A, B, and $C=\lambda A+(1-\lambda)B$, for $0 \leqq \lambda \leqq 1$, be three positive definite matrices with the respective sets of characteristic values (which we take to be arranged in increasing order), x_i, y_i, z_i. Let us define

$$|A|_k = \prod_{i=1}^{k} x_i, \qquad |B|_k = \prod_{i=1}^{k} y_i, \qquad |C|_k = \prod_{i=1}^{k} z_i. \tag{1.1}$$

Then Ky Fan showed, [1] (see also [2], [3] for an alternate proof), that

$$|C|_k \geqq |A|_k^{\lambda} |B|_k^{(1-\lambda)}. \tag{1.2}$$

In another paper, [4], he showed that the following inequality is also valid:

$$\sum_{i=1}^{k} z_i \geqq \lambda \sum_{i=1}^{k} x_i + (1-\lambda) \sum_{i=1}^{k} y_i. \tag{1.3}$$

Let us now show that (1.3) is a consequence of (1.2).

2. Proof. Let us define three new positive definite matrices by means of the relations

$$X = I + \epsilon A, \qquad Y = I + \epsilon B, \qquad Z = I + \epsilon C, \tag{2.1}$$

where ϵ is a small positive quantity. The characteristic roots of X are $1+\epsilon x_i$, those of Y are $1+\epsilon y_i$, while those of Z are $1+\epsilon z_i$. Substituting in the relation in (2), we have, keeping only terms of the zero-th and first degree in ϵ,

$$1 + \epsilon\left(\sum_{i=1}^{k} z_i\right) \geqq \left(1 + \epsilon \sum_{i=1}^{k} x_i\right)^{\lambda}\left(1 + \epsilon \sum_{i=1}^{k} y_i\right)^{(1-\lambda)} + o(\epsilon) \tag{2.2}$$

or

$$1 + \epsilon\left(\sum_{i=1}^{k} z_i\right) \geqq \left(1 + \epsilon\lambda \sum_{i=1}^{k} x_i\right)\left(1 + \epsilon(1-\lambda) \sum_{i=1}^{k} y_i\right) + o(\epsilon) \tag{2.3}$$

or

$$\sum_{i=1}^{k} z_i \geqq \lambda \sum_{i=1}^{k} x_i + (1-\lambda) \sum_{i=1}^{k} y_i + o(1). \tag{2.4}$$

Allowing ϵ to approach zero, we obtain (1.3).

3. Remarks (suggested by referee). The inequality above is true for arbitrary symmetric matrices since we can always find a positive scalar c with the property that $A+cI$ and $B+cI$ are positive definite. From this it follows that we can formulate the result in (1.3) in a manner which is independent of λ: If A, B, $C=A+B$ are three symmetric matrices with the respective sets of characteristic values, arranged in increasing order, x_i, y_i, z_i, then

$$\sum_{i=1}^{k} z_i \geqq \sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i. \tag{3.1}$$

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2. ———, I. Glicksberg, and O. Gross, Notes on matrix theory—VI, this MONTHLY, vol 62, 1955, pp. 571–572.
3. Ky Fan, Problem 4430, this MONTHLY, vol. 58, 1951, pp. 194; solution in vol. 60, 1953, p. 50.
4. ———, On a theorem of Weyl concerning eigenvalues of linear transformations—I, *Proc. Nat. Acad. Sci.*, vol. 35, 1949, p. 65.

THE BLASIUS FORMULAE

G. Power, University of Nottingham

In recent years, a great deal of consideration has been given to flow problems, and in particular to the force experienced by a cylinder placed in a two-dimensional field. Emphasis has been placed on the well-known Blasius formulae when a complex potential function exists (as a function of z only), especially in relation to the cases of a cylinder in an incompressible inviscid fluid, and a cylindrical conductor in an electrical field. It is perhaps not generally realized that formulae of this type can be obtained for the force caused by the refraction of the lines of flow at a surface of discontinuity separating different dielectric media, without assuming the existence of a complex potential function for the electric field.

Consider a nested system of homogeneous isotropic dielectric cylinders having a cross-section of general shape, and let us examine the stress on the contour of the interface separating the medium (s) of dielectric constant k_s from the medium $(s+1)$ of dielectric constant k_{s+1}. Take the origin of coordinates inside the contour and let a pair of rectangular axes define the complex z-plane in the usual manner.

If X, Y are the components of electric intensity due to the field in the medium $(s+1)$ then the Maxwell stress components are:

$$P_{xx} = \frac{k_{s+1}}{8\pi}\{X^2 - Y^2\}, \qquad P_{yy} = \frac{k_{s+1}}{8\pi}\{Y^2 - X^2\}, \qquad P_{xy} = P_{yx} = \frac{k_{s+1}}{4\pi}XY.$$

The effects of electrostriction have been ignored. The interface C will experience a force whose components per unit length are

$$X_{s+1} = \int_C (P_{xx}dy - P_{xy}dx), \qquad Y_{s+1} = \int_C (P_{yx}dy - P_{yy}dx),$$

so that the complex force is

$$Z_{s+1} = Y_{s+1} + iX_{s+1} = \frac{k_{s+1}}{8\pi}\int_C (X - iY)^2 dz. \tag{1}$$

In a similar manner Γ_{s+1}, the anticlockwise moment about the origin, is given by

$$\Gamma_{s+1} = Re\frac{k_{s+1}}{8\pi}\int_C (X - iY)^2 z dz. \tag{2}$$

There are similar expressions for the complex force Z_s and couple Γ_s due to the field in medium (s). The total mechanical force and couple on the interface are thus given respectively by

$$Z_{s,s+1} = Z_{s+1} - Z_s, \qquad \Gamma_{s,s+1} = \Gamma_{s+1} - \Gamma_s. \tag{3}$$

These formulae do not take into account the tendency of the dielectric material to change its shape. In general it is impossible to include this effect for two solid media, but it can be demonstrated that if the medium $(s+1)$ is fluid, electrostriction is accounted for by omitting the last term on the right-hand side of each of the equations (3).

The corresponding results for a cylinder in a steady incompressible inviscid fluid flow are

$$(4) \qquad Z = -\frac{\rho}{2}\int_C (u - iv)^2 dz, \qquad \Gamma = Re - \frac{\rho}{2}\int_C (u - iv)^2 z dz,$$

where u, v are the fluid velocity components. We note that the motion need not be irrotational. It should be noted that in the above results the integrands are not analytic functions of z, but this difficulty disappears since on the interface C $\bar{z}$, the complex conjugate of z, is known in terms of z.

Of course, when there is a complex potential as a function of z only, all the above formulae take the normal Blasius form, which is obviously much more useful mathematically.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

REMARK ON EQUIVALENCE RELATIONS

R. A. ROSENBAUM, Wesleyan University

Students are sometimes bothered by the following pseudo-proof that symmetry and transitivity together imply reflexivity:

Suppose that, for arbitrary x, $x \sim y$. If $y = x$, there is nothing to prove. If $y \neq x$, by symmetry, $y \sim x$. Hence, by transitivity, $x \sim x$.

The difficulty, of course, lies in the assumption of the existence of y such that $x \sim y$. The following example may be of use in clarifying the situation:

Let $S = \{1, 2, 4, 6, 10\}$, and let $p \sim q$ be interpreted as meaning $p + q \equiv 0 \pmod 4$. It is immediate from the definition that the relation is symmetric and it is easy to verify that it is transitive. We observe that $2 \sim 6$, and hence expect, by the argument of the pseudo-proof, that $2 \sim 2$, which is the case. We note that there is no $x \neq 1$ such that $x \sim 1$. Therefore, we cannot prove that $1 \sim 1$; and, indeed $1 \nsim 1$. Finally, although there is no $x \neq 4$ such that $x \sim 4$, it is true that $4 \sim 4$.

THE CALCULUS OF ABSOLUTE VALUES

KENNETH O. MAY, Carleton College

The customary way of dealing with functions involving absolute values is to consider separately the cases in which the expressions within absolute value symbols are positive and negative. This is clumsy, particularly where differentiation or integration is involved. It may be responsible for the neglect of such functions, even when they would be appropriate in applications. The fact is that the function defined by $y=|x|$ is a very well behaved continuous function with a continuous derivative everywhere except at the origin. There exist simple formulas by which it may be differentiated and integrated with no more attention to discontinuities or existence than is required for such functions as the negative powers of x.

A useful identity is

$$|x| = x\,\mathrm{sg}x \tag{1}$$

where sg is the signum function defined by $\mathrm{sg}x=-1, 0, 1$ according as $x<, =, >0$. Evidently $D\mathrm{sg}x=0$ everywhere except at the origin, where $\mathrm{sg}x$ has a jump of 2. We note also that $\mathrm{sg}x=|x|/x=x/|x|$ for $x\neq 0$. Then it is easy to establish that

$$D|x| = |x|/x. \tag{2}$$

We have also, for $x\neq 0$,

$$D|x| = \mathrm{sg}x. \tag{3}$$

However, (2) has the advantage of calling attention to the non-existence of the derivative for $x=0$.

With the aid of (2) and the chain rule we can find derivatives of composite functions involving absolute values. By this means it is easy to show that

$$D|x^n| = nx^{n-1}\mathrm{sg}x^n. \tag{4}$$

This holds for all x if $n>1$. If $n\leqq 1$, the derivative fails to exist at the origin. More generally,

$$D^n|f(x)| = f^{(n)}(x)\mathrm{sg}f(x) \tag{5}$$

wherever the derivative on the left exists.

The maxima and minima of $|f(x)|$ and $f(x)$ are related in a simple way. Obviously $|f(x)|$ has an absolute minimum at any zero of $f(x)$. Since in any interval where $f(x)>0$ the graphs of $|f(x)|$ and $f(x)$ coincide, they have relative maxima and minima at the same points in such intervals. In intervals where $f(x)<0$ the graphs of $|f(x)|$ and $f(x)$ are each other's mirror images in the x-axis, and $|f(x)|$ has a relative maximum (minimum) where $f(x)$ has a relative minimum (maximum). These remarks hold even if derivatives do not exist at the points in question. However, if the derivative tests are applicable, formula (5)

gives the same results.

Integration by parts yields the following formula, and it can easily be verified by differentiating the right member.

$$\int |x|\,dx = \tfrac{1}{2}x|x| = \tfrac{1}{2}x^2\operatorname{sg}x. \tag{6}$$

Similarly one can find and verify

$$\int |x^n|\,dx = \frac{1}{n+1}|x^n|x = \frac{1}{n+1}x^{n+1}\operatorname{sg}x^n \qquad (n \neq -1). \tag{7}$$

As an application of some of the above ideas, consider the problem of proving that a median of a probability distribution is a point from which the mean of the absolute deviations is a minimum. We consider only the case of a continuous distribution, though the discrete case or the general case may be treated by the same methods. The problem is to minimize the integral

$$F(t) = \int_{-\infty}^{\infty} |x - t|\,f(x)dx \tag{8}$$

where $f(x)$ is non-negative and the integral of $f(x)$ from $-\infty$ to ∞ is 1. The usual proof is rather clumsy. However, using the above formulas, we have

$$F'(t) = \int_{-\infty}^{\infty} \operatorname{sg}(t - x)f(x)dx \tag{9}$$

$$= \int_{-\infty}^{t} f(x)dx - \int_{t}^{\infty} f(x)dx \tag{10}$$

$$= 2\int_{-\infty}^{t} f(x)dx - 1. \tag{11}$$

But a median is defined as a value of t that makes the right member of (11) zero. Moreover, since

$$F''(t) = 2f(t) \geqq 0 \tag{12}$$

for all values of t, $F'(t)$ is monotonically increasing, and equation (11) does yield an absolute minimum of $F(t)$.

As a second example, consider the following problem given in Kaplan's *Advanced Calculus* on page 515. Where is the function $|x^2-y^2|+i|2xy|$ analytic? Since $u=|x^2-y^2|$ and $v=|2xy|$ are continuous functions with continuous derivatives in any domain not including points on the axes or on the lines defined by $x^2=y^2$, we need consider only the Cauchy-Riemann conditions. We have

$$\frac{\partial u}{\partial x} = 2x\operatorname{sg}(x^2 - y^2) \qquad \frac{\partial v}{\partial y} = 2x\operatorname{sg}(xy)$$

$$\frac{\partial u}{\partial y} = -2y\,\mathrm{sg}(x^2-y^2) \quad -\frac{\partial v}{\partial x} = -2y\,\mathrm{sg}(xy).$$

The condition for analyticity is evidently $\mathrm{sg}(x^2-y^2)=\mathrm{sg}(xy)$, and hence the domains are defined by $xy(x^2-y^2)>0$. The procedure is much simpler than considering the combinations of cases for different signs of x^2-y^2 and $2xy$.

EXPANSIONS FOR π AND π^2

L. L. PENNISI, University of Illinois

By establishing the following two formulas, we shall be able to give several expansions for π and π^2 in terms of infinite series.

If $|u|<1$, then

$$(1) \qquad \sum_{n=0}^{\infty} \frac{n!2^n u^{2n}}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n+1)} = \frac{\arcsin u}{u\sqrt{1-u^2}}$$

and

$$(2) \qquad \frac{u^2}{2} + \sum_{n=1}^{\infty} \frac{2\cdot 4\cdot\cdots\cdot 2nu^{2n+2}}{[1\cdot 3\cdot 5\cdot\cdots\cdot(2n+1)](2n+2)} = \tfrac{1}{2}[\arcsin u]^2.$$

Proof. By a well known formula (see, A. Zygmund, Trigonometrical Series, p. 222), we have

$$\frac{1}{n!}\int_0^{1/2}\left(\frac{1}{2}-t\right)^n (2t)^{-1/2}dt = \int_0^{1/2} dx_{n+1}\int_0^{x_{n+1}} dx_n\cdots\int_0^{x_3} dx_2\int_0^{x_2}(2x_1)^{-1/2}dx_1$$

$$= \frac{1}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n+1)}.$$

Hence

$$\sum_{n=0}^{\infty}\frac{n!(2u^2)^n}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n+1)} = \sum_{n=0}^{\infty}\int_0^{1/2}\left(\frac{1}{2}-t\right)^n (2u^2)^n(2t)^{-1/2}dt.$$

As a consequence of the Lebesgue Monotone Convergence Theorem (see, M. E. Munroe, Introduction to Measure and Integration, pp. 186–187), we have

$$\sum_{n=0}^{\infty}\int_0^{1/2}\left(\frac{1}{2}-t\right)^n (2u^2)^n(2t)^{-1/2}dt$$

$$= \int_0^{1/2}\sum_{n=0}^{\infty}\left[\left(\frac{1}{2}-t\right)2u^2\right]^n (2t)^{-1/2}dt$$

$$= \int_0^{1/2}\frac{(2t)^{-1/2}dt}{(1-u^2)+2u^2t} = \frac{1}{u\sqrt{1-u^2}}\arctan\frac{u}{\sqrt{1-u^2}} = \frac{\arcsin u}{u\sqrt{1-u^2}},$$

thus establishing (1). By integrating (1), we obtain (2).

Example 1. Substituting $u=1/\sqrt{2}$ into (1), we obtain

$$1+\frac{1}{3}+\frac{2!}{3\cdot 5}+\frac{3!}{3\cdot 5\cdot 7}+\cdots=\frac{\pi}{2}.$$

Example 2. Substituting $u=\frac{1}{2}$ into (2), we obtain

$$\frac{1}{8}+\sum_{n=1}^{\infty}\frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots(2n+1)(2n+2)2^{2n+2}}=\frac{\pi^2}{72}.$$

A THEOREM ON THE TAYLOR EXPANSION

C. S. OGILVY, Hamilton College

Consider the Taylor expansion of $f(x+h, y+k)$ around the point (x, y):

$$f(x+h, y+k)=f(x, y)+\sum_{n=1}^{\infty}\phi_n,$$

where

$$\phi_n=1/n!\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x, y),$$

and

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x, y)$$

means

$$h^2\frac{\partial^2 f(x, y)}{\partial x^2}+2hk\frac{\partial^2 f(x, y)}{\partial x\partial y}+k^2\frac{\partial^2 f(x, y)}{\partial y^2}$$

and similarly with the higher order terms.

Suppose we set $\phi_n=0$, $n=2, 3, \cdots$. Then we have the

THEOREM: *If there exists an expression (in x, y) for h/k which annuls identically both ϕ_2 and ϕ_3, then it annuls identically also all ϕ_n, $n>3$.*

A geometric proof is easy. The elimination of h/k between $\phi_2=0$ and $\phi_3=0$ results in the well known partial differential equation of the general *ruled surface.** This is another way of saying that if there exists a function h/k which satisfies both $\phi_2=0$ and $\phi_3=0$, then $f(x, y)$ represents a ruled surface. We have previously shown† that the existence of an h/k such that $f(x+h, y+k)=f(x, y)+\phi_1$ is a necessary and sufficient condition for the surface $f(x, y)$ to be ruled. This is obviously the same h/k which annuls ϕ_2 and ϕ_3; and hence it annuls all higher order terms also.

The theorem may be of value in non-geometric applications. A purely analytic proof is feasible, but the required induction appears to be tedious.

* Goursat-Hedrick, Vol. II, Part II, p. 281, equations (A) and (B).

† C. S. Ogilvy, This MONTHLY, vol. 59, 1952, p. 548, equation (1).

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1186. *Proposed by David Freedman, Student, McGill University*

Show that the total number of permutations of n things is $[n!e]$, where $[x]$ denotes the greatest integer in x.

E 1187. *Proposed by James Goode, Student, Georgia Institute of Technology*

Find

$$\lim_{n\to\infty} \sum_{k=n}^{2n} 1/k.$$

E 1188. *Proposed by Alice Anderson, Student, East Tennessee State College*

Each card used in the game of Bingo is made up of a square array of 25 numbers. The five numbers in the first column are selected from 1 to 15, those in the second column from 16 to 30, those in the third column from 31 to 45, those in the fourth column from 46 to 60, and those in the fifth column from 61 to 75. The space common to the third column and third row is left blank. If no card can have a row, column, or diagonal identical to a row, column, or diagonal of another card, how many different cards can be made?

E 1189. *Proposed by N. A. Court, University of Oklahoma*

If two pairs of spheres with non-coplanar centers are such that each sphere of one pair is orthogonal to the two spheres of the other pair, then the tetrahedron formed by the centers of similitude of the two pairs of spheres is orthocentric.

E 1190. *Proposed by R. A. Miller, University of Mississippi*

A student wrote the following in solving the problem, "Find the value of $(3+x)^{2/3}$ when $x=3/8$:"

$$(3 + x)^{2/3} = (3 + 3/8)^{2/3} = 2 + 1/4 = 9/4.$$

Investigate the conditions under which

$$(a + b/c)^{m/n} = (a + b/c)(m/n),$$

where a, b, c, m, n are positive integers.

SOLUTIONS

A Minimal Collection of Positive Integers

E 1157 [1955, 181]. *Proposed by T. F. Mulcrone, St. Charles College, Grand Coteau, La.*

Determine the smallest collection of positive integers such that for any integer k, $0<k\leqq c$, where c is a fixed integer, there is a subcollection whose sum is k.

Solution by L. A. Ringenberg, Eastern Illinois State College. Since c different sums of subcollections are required, it follows that the required collection has at least c distinct subcollections (excluding the null set as a subcollection). If n is the cardinal number of the required smallest collection and if m is the smallest integer such that $2^m-1\geqq c$, then $n\geqq m$. But every k, $0<k\leqq c$, is the sum of a subcollection of the collection $1, 2, 4, \cdots, 2^{m-1}$. Hence $n\leqq m$ and, finally, $n=m$.

Also solved by Hüseyin Demir, A. J. Goldman, Virginia Hanly, A. R. Hyde, J. M. Kingston, M. S. Klamkin, D. C. B. Marsh, J. V. Pennington, Azriel Rosenfeld, Michael Skalskyj, D. D. Strebe, Alan Wayne, and the proposer.

Several solvers saw in this the Bachet weights problem for the "one-pan" case. Wayne called attention to MacMahon's article in *Quarterly Journal of Mathematics*, vol. 21, 1886, pp. 367–373.

Roots of a Polynomial

E 1158 [1955, 181]. *Proposed by J. E. Hanson, Johns Hopkins University*

Find the roots of

$$\sum_{i=0}^{n}\binom{n+i+1}{2i+1}x^i=0, \qquad n\geqq 1.$$

Solution by James Singer, Brooklyn College. Put

$$p_0(x)=1,$$

$$p_n(x)=\sum_{i=1}^{n}\binom{n+i+1}{2i+1}x^i, \qquad n\geqq 1.$$

It is readily proved that

$$p_{n+2}(x)-(2+x)p_{n+1}(x)+p_n(x)\equiv 0, \qquad n\geqq 0.$$

By use of the method of difference equations, we find

$$p_n(x)=(\alpha^{n+1}-\beta^{n+1})/(\alpha-\beta),$$

where α and β are the roots of $y^2-(x+2)y+1=0$. Hence a root of $\alpha=\omega\beta$, where $\omega^{n+1}=1$, $\omega\neq 1$, is a root of $p_n(x)=0$. A calculation yields the n (distinct, real, negative) roots of $p_n(x)=0$, namely

$$x=2\left(\cos\frac{k\pi}{n+1}-1\right), \qquad k=1, 2, \cdots, n.$$

Also solved by W. J. Blundon, Leonard Carlitz, Chih-yi Wang, and the proposer.

Concerning the Euclidean Algorithm

E 1159 [1955, 181]. *Proposed by Albert Newhouse, University of Houston*

In finding the greatest common divisor (a, b) of a and b (either integers or polynomials) by the Euclidean algorithm there appear, in the successive steps, quotients q_i and remainders r_i. Show that $(a, b) = as + bt$, where

$$s = (-1)^{k-1}\begin{vmatrix} q_2 & 1 & 0 \cdots 0 \\ -1 & q_3 & 1 \cdots 0 \\ 0 & -1 & q_4 \cdots 0 \\ \cdot & \cdot & \cdot \cdots \cdot \\ 0 & 0 & 0 \cdots q_k \end{vmatrix}, \quad t = (-1)^k\begin{vmatrix} q_1 & 1 & 0 \cdots 0 \\ -1 & q_2 & 1 \cdots 0 \\ 0 & -1 & q_3 \cdots 0 \\ \cdot & \cdot & \cdot \cdots \cdot \\ 0 & 0 & 0 \cdots q_k \end{vmatrix}.$$

I. *Solution by A. R. Hyde, West Hartford, Conn.* It can be shown that each $r_i = as_i + bt_i$, where s_i and t_i are correspondingly defined. This is easily verified for $i = 1$ (if s_1 is defined to be 1) and for $i = 2$. Suppose it to hold for $i-1$ and i. By the algorithm,

$$r_{i+1} = r_{i-1} - q_{i+1}r_i = a(s_{i-1} - q_{i+1}s_i) + b(t_{i-1} - q_{i+1}t_i).$$

If the s_{i+1} determinant is expanded with respect to the last row (or column), it is seen that $s_{i+1} = s_{i-1} - q_{i+1}s_i$; similarly $t_{i+1} = t_{i-1} - q_{i+1}t_i$. Hence $r_{i+1} = as_{i+1} + bt_{i+1}$, and the desired relation holds for $i+1$. Therefore the relation holds to the end of the algorithm.

II. *Solution by Chih-yi Wang and David Zeitlin, University of Minnesota.* The successive steps in the Euclidean algorithm can be considered as a system of $k+1$ linear equations in the k unknowns $r_1, r_2, \cdots, r_k$. Since the system is consistent we may consider only the first k equations. Solving these for r_k by Cramer's rule easily yields the desired result.

Also solved by Hüseyin Demir, A. J. Goldman, William Googe, M. S. Klamkin, D. C. B. Marsh, Azriel Rosenfeld, Michael Skalskyj, D. D. Strebe, Lincoln Teng, and the proposer.

Editorial Note. It is interesting to note, using the notation of the first solution, that $as_{k+1} + bt_{k+1} = 0$ and that

$$(a, b) = a/(-1)^k s_{k+1} = b/(-1)^{k+1}t_{k+1}.$$

In this connection see the following papers by E. Frank: (1) *On the zeros of polynomials with complex coefficients*, Bull. Amer. Math. Soc., vol. 52, 1946, pp. 144–158, (2) *The location of the zeros of polynomials with complex coefficients*, Bull. Amer. Math. Soc., vol. 52, 1946, pp. 890–898, (3) *On certain determinantal equations*, this MONTHLY, vol. 59, 1952, pp. 300–309.

A Property of the Newton Line of a Complete Quadrilateral

E 1160 [1955, 182]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Prove that in a complete quadrilateral the isotomic line of any side with respect to the triangle formed by the other three is parallel to the Newton line of the quadrilateral.

I. *Solution by the Proposer.* Let d be one of the four sides of the quadrilateral and let ABC be the corresponding triangle. Denote the intersections of d with the sides BC, CA, AB of triangle ABC by α, β, γ. The isotomic line IJK of $\alpha\beta\gamma$ with respect to triangle ABC is obtained by taking the symmetrics I, J, K of the points α, β, γ with respect to the midpoints A', B', C' of the sides BC, CA, AB of triangle ABC. Let the midpoints of $A\alpha$, $B\beta$, $C\gamma$ be denoted by I', J', K'. These points of the Newton line of the quadrilateral are evidently on the sides of the medial triangle $A'B'C'$ of triangle ABC. It is easy to see that the complete quadrilateral formed by triangle ABC and line IJK is similar to that formed by triangle $A'B'C'$ and line $I'J'K'$, for, firstly, triangles ABC and $A'B'C'$ are similar, and are in the ratio 2:1, and secondly,

$$BI = C\alpha = 2(B'I'), \qquad AJ = C\beta = 2(A'J'), \qquad AK = B\gamma = 2(A'K').$$

This proves that the lines IJK and $I'J'K'$ are parallel.

II. *Solution by Sister M. Stephanie, Georgian Court College, Lakewood, N.J.* Since there is one and only one parabola tangent to four lines, let us consider the complete quadrilateral as tangent to the parabola (referred to rectangular coordinates) $y^2 = 4ax$. Then $y = m_i x + a/m_i$, $i = 1, 2, 3, 4$, may be taken as the equations of the four sides 1, 2, 3, 4 of the quadrilateral. Point

$$(a/m_1m_2,\ a/m_1 + a/m_2)$$

is the intersection of sides 1 and 2; other intersections are similarly given. The midpoint of the side 2 of triangle 123 has coordinates

$$(a/2)[(m_1 + m_3)/m_1m_2m_3,\ 2/m_2 + 1/m_1 + 1/m_3].$$

If (x, y) is the point on side 2 isotomic to the intersection of side 4 with side 2, then

$$y + a/m_2 + a/m_4 = 2a/m_2 + a/m_1 + a/m_3,$$

whence

$$y = a(1/m_1 + 1/m_2 + 1/m_3 - 1/m_4),$$

a result which is symmetric in m_1, m_2, m_3. This proves that the isotomic line of side 4 with respect to triangle 123 is parallel to the axis of the parabola. But the Newton line is also parallel to the axis of the parabola, for it is the locus of centers of all conics inscribed in the quadrilateral, and this locus contains the center of the parabola.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4658. *Proposed by P. R. Halmos, University of Chicago*

If X is a set, let X^2 be the Cartesian product of X with itself. Call a subset D of X^2 a *diagonal* if for every x in X there exists a unique y in X and there exists a unique z in X such that $(x, y)\in D$ and $(z, x)\in D$. Prove that there exists a mapping from X^3 onto X such that the inverse image of every point is a diagonal.

4659. *Proposed by R. C. Lyness, Preston, England*

A solid of uniform density is bounded by those halves of the surfaces $x^2+y^2+z^2=a^2$ and $x^2+y^2+c^2z^2=a^2$ for which $z\geqq 0$. Find c so that under an inverse square law the solid attracts a particle at the origin as if its mass were concentrated at a point on its surface.

4660. *Proposed by E. M. Wright, University of Aberdeen, Scotland*

For all $x\geqq 1$, $f(x)$ and $\phi(x)$ are non-negative functions, bounded and integrable in any finite interval. They satisfy the inequality

$$xf(x) \leqq \int_1^x f(t)dt + \phi(x).$$

(i) If $\int^\infty \phi(t)t^{-2}dt = \infty$, find an $f(x)$ such that $f(x)\to\infty$ as $x\to\infty$.

(ii) If $\phi(x)/x\to 0$ and $\int^\infty \phi(t)t^{-2}dt < \infty$. Show that

$$g(x) = \frac{1}{x}\int_1^x f(t)dt$$

tends to a limit as $x\to\infty$ and that $\lim g(x) = \overline{\lim} f(x)$.

(iii) Show that, whatever restriction we may impose on the order of $\phi(x)$ as $x\to\infty$, we cannot thereby ensure that $f(x)$ tends to a limit.

4661. *Proposed by J. B. Kelly, Michigan State University*

From an urn containing m black balls and m white balls, balls are drawn one at a time without replacement. An observer guesses the outcome (black or white) of each drawing. It is assumed that at any stage he will guess black if there remain more black balls than white ones and vice versa. Determine $E(m)$

the expected value of the number of correct guesses made during the entire procedure until all the balls have been withdrawn. Give an asymptotic formula for $E(m)-m$.

4662. *Proposed by J. E. Wilkins, Jr., Nuclear Development Corporation of America, White Plains, N. Y.*

If X is a measure space and if $\phi_i(x)$ and $\psi_j(x)$ are two sequences of functions in $L_2(X)$ which are complete in $L_2(X)$, then the sequence $\phi_i(x)\psi_j(y)$ is complete in $L_2(X, X)$.

SOLUTIONS

Stirling Numbers of the Second Kind

4595 [1954, 427]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York*

If

$$[xD]^n = \sum_{r=1}^{n} A_{r,n} x^{n-r+1} D^{n-r+1},$$

where D is the differential operator, determine $A_{r,n}$.

Solution by A. S. Hendler, Rensselaer Polytechnic Institute, Troy, N. Y. On writing $[xD]^n = x \cdot D[xD]^{n-1}$, we see that the $A_{r,n}$ must satisfy the equation of partial differences

$$A_{r,n} = A_{r,n-1} + (n - r + 1)A_{r-1,n-1},$$

and the initial conditions are $A_{1,n} = A_{n,n} = 1$. Thus the $A_{r,n}$ are completely determined.

However, if we take $A_{n-k+1,n} = B_{k,n} (k = 1, 2, \cdots, n)$ the equation of partial differences may be written

$$B_{k,n} = B_{k-1,n-1} + kB_{k,n-1}$$

with the initial conditions $B_{n,n} = B_{1,n} = 1$. But $\mathfrak{S}_n^1 = \mathfrak{S}_n^n = 1$, where $\mathfrak{S}_n^k$ are the Stirling numbers of the second kind defined by

$$\mathfrak{S}_n^k = \frac{1}{k!} \sum_{i=1}^{k} (-1)^{k-i} {}_kC_i i^n.$$

Also an easy calculation will verify that

$$\mathfrak{S}_{n-1}^{k-1} + k\mathfrak{S}_{n-1}^{k} = \mathfrak{S}_n^k.$$

Hence $B_{k,n} = \mathfrak{S}_n^k$ and

$$A_{r,n} = \mathfrak{S}_n^{n-r+1} = \frac{1}{(n-r+1)!} \sum_{i=1}^{n-r+1} (-1)^{n-r+1-i} {}_{n-r+1}C_i i^n.$$

Also solved by N. Balasubramanian, W. J. Blundon, L. Carlitz, H. W. Gould, D. S. Greenstein, A. T. Hind, Jr., A. R. Hyde, D. C. B. Marsh, Norman Miller, Leo Moser, S. Parameswaran, Michael Skalskyj, M. R. Spiegel, Chih-yi Wang, and the Proposer.

Editorial Note. The problem has been treated in a number of places and the following references were cited by our correspondents: Jordan, *Calculus of Finite Differences*, pp. 195–196, (See also pp. 168–170 for the Stirling numbers), Schwatt, *Introduction of Operations with Series* (1924), pp. 86 ff., and an article by L. Carlitz, *On a class of finite sums*, this MONTHLY, (1930) pp. 473–479.

Almost Absolutely Convergent Sequences

4596 [1954, 428]. *Proposed by G. G. Lorentz, Wayne University, and M. S. Macphail, Carleton College, Ottawa*

Given a sequence s_n, let

$$\sigma_{n,p} = \frac{s_n + s_{n+1} + \cdots + s_{n+p-1}}{p}. \tag{1}$$

We call s_n almost absolutely convergent (a.a.c.) if the series

$$\sum_{p=1}^{\infty} |\sigma_{n,p} - \sigma_{n,p+1}|$$

converges uniformly for $n = 1, 2, \cdots$. Prove that:

1. A sequence s_n is a.a.c. if s_n is absolutely convergent, that is if $\sum |s_n - s_{n+1}| < \infty$.

2. If each a.a.c. sequence is absolutely summable by a method of summation $A = (a_{mn})$, then there is a bounded sequence which is absolutely A summable but is not a.a.c.

Solution by the Proposers.

1. From (1) we have

$$\sigma_{n,p} - \sigma_{n,p+1} = \frac{1}{p(p+1)}\left[(s_n - s_{n+1}) + 2(s_{n+1} - s_{n+2}) + \cdots + p(s_{n+p-1} - s_{n+p})\right].$$

Taking absolute values and summing with respect to p,

$$\sum_{p \geqq p_0} |\sigma_{n,p} - \sigma_{n,p+1}| \leqq \sum_{r=1}^{\infty} |s_{n+r} - s_{n+r+1}| = R_n.$$

It follows that all series

$$\sum_{p=1}^{\infty} |\sigma_{n,p} - \sigma_{n,p+1}|, \qquad n = 1, 2, \cdots \tag{2}$$

are convergent. Let $\epsilon>0$ be arbitrary, and take n_0 so large that $R_n<\epsilon$ for $n\geqq n_0$. Then we take p_0 such that

$$\sum_{p\geqq p_0} |\sigma_{n,p}-\sigma_{n,p+1}| < \epsilon \qquad \text{for } n = 1, 2, \cdots, n_0. \tag{3}$$

The preceding inequalities then show that (3) holds for all n.

2. We need a Lemma. Let $\Omega(n)$ be a positive increasing function with $\Omega(n)\to\infty$ and

$$\sum_{n=1}^{\infty} \Omega(n)n^{-2} < \infty \tag{4}$$

(for instance, let $\Omega(n)=n^{1/2}$.) Let the sequence $n_1<n_2<\cdots$ of integers increase so rapidly that its counting function $\omega(n)$ ($\omega(n)$ is the number of ν for which $n_\nu\leqq n$) satisfies

$$\omega(n+m)-\omega(n)\leqq\Omega(m) \qquad n, m = 1, 2, \cdots. \tag{5}$$

Let s_n be a bounded sequence such that $s_n=0$ if n is not one of the indices n_ν. Then s_n is a.a.c.

For the proof we show that the series $\sum_{n_\nu>n}(n_\nu-n)^{-1}$, $n=1, 2, \cdots$, are uniformly convergent for all n. In fact, the rest of the above series corresponding to $\nu\geqq\nu_0$, is

$$\begin{aligned}\sum_{\nu=\nu_0}^{\infty}(n_\nu-n)^{-1} &\leqq C\sum_{\nu=\nu_0}^{\infty}\sum_{m\geqq n_\nu-n} m^{-2} = C\sum_{m=n_{\nu_0}}^{\infty} m^{-2}\sum_{n<n_\nu\leqq n+m} 1\\ &= C\sum_{m=n_{\nu_0}}^{\infty} m^{-2}[\omega(n+m)-\omega(n)]\\ &\leqq C\sum_{m=n_{\nu_0}}^{\infty}\Omega(m)m^{-2},\end{aligned}$$

and the last expression is small for large ν_0 and does not depend on n.

Returning to s_n, assume first that $n+p=n_\nu$, then if $|s_n|\leqq M$,

$$\begin{aligned}|\sigma_{n,p}-\sigma_{n,p+1}| &= \left|\frac{1}{p(p+1)}(s_n+\cdots+s_{n+p-1})-\frac{1}{p+1}s_{n+p}\right|\\ &\leqq \frac{1}{p(p+1)}\Omega(p)M+\frac{1}{p+1}M \leqq M(p^{-2}\Omega(p)+(n_\nu-n)^{-1}).\end{aligned} \tag{6}$$

If $n+p$ is not a n_ν, we obtain in the same way

$$|\sigma_{n,p}-\sigma_{n,p+1}| \leqq Mp^{-2}\Omega(p). \tag{7}$$

Relations (6) and (7) prove the Lemma.

Considering now the method $A'=(a_{m,n_\nu})$, $m, \nu=1, 2, \cdots$, we see that A' sums absolutely each bounded sequence and hence (by G. G. Lorentz, *Direct*

theorems on methods of summability II, Canadian Journ. Math. 3 (1951), pp. 236–256, Theorem 7)

$$(8) \qquad V(n_\nu) = \operatorname*{Var}_m a_{m,n_\nu} = \sum_{m=1}^{\infty} |a_{m,n_\nu} - a_{m+1,n_\nu}| \to 0 \qquad \text{for } \nu \to \infty.$$

Since n_ν was an arbitrary sequence increasing sufficiently rapidly, (8) implies that

$$V(n) = \operatorname*{Var}_m a_{mn} \to 0 \qquad \text{for } n \to \infty.$$

Now it is sufficient to take a sequence s_n of the form

$$1, 0, \cdots, 0, 1, 1, 0, \cdots, 0, 1, 1, 1, 0, \cdots$$

with arbitrary long blocks of 1's separated by long blocks of 0's such that $\sum V(n_\nu) < \infty$, where the n_ν correspond to the values 1 of s_n. Then s_n is absolutely A summable, but is not a.a.c., which proves the assertion.

Representations in Terms of a Given Sequence

4597 [1954, 428]. *Proposed by Paul Erdös, University of Notre Dame*

Let $1 = a_1 < a_2 < \cdots$ be an infinite sequence of integers. Let n be any integer and write $n = a_{i_1} + a_{i_2} + \cdots + a_{i_k}$ where a_{i_k} is the greatest $a \leqq n$, a_{i_2} the greatest $a \leqq n - a_{i_1}$, *etc.* (Since $a_1 = 1$ this representation is always possible.) Put $f(n) = k$ (*i.e.*, the number of summands representing n). Prove that if the upper density of the a's is 0, then

$$\lim_{x\to\infty} \frac{1}{x} \sum_{k=1}^{x} f(k) = \infty,$$

and if the lower density of the a's is >0, then

$$\overline{\lim_{x\to\infty}} \frac{1}{x} \sum_{k=1}^{x} f(k) < \infty.$$

(If $N(y)$ denotes the number of a's $\leqq y$, the upper density of the a's is $\overline{\lim}\, N(y)/y$, and the lower density is $\underline{\lim}\, N(y)/y$.)

Solution by Howard Robbins, Mantorville, Minnesota. An identity for $S(x) = \sum_{r=1}^{x} f(r)$ can be derived by separating out the first term in the a-expansion of each integer r:

$$S(x) = x + S(x - a_n) + \sum_{k=2}^{n} S(a_k - a_{k-1} - 1).$$

Here a_n is the largest $a \leqq x$. There are x terms separated out, $S(a_k - a_{k-1} - 1)$ gives the additional terms needed to expand the integers $a_{k-1}+1$ through $a_k - 1$, and $S(x - a_n)$ gives the additional terms needed for integers $a_n + 1$ through x.

If the lower density of the a's is not zero, a number $M>1$ can be found such that $M \cdot N(x) \geqq x$ for all x. Then by induction it can be shown that $S(x) < Mx$ for all x, establishing one of the two desired results: The relation holds for $x=0$ or 1, and if it holds for all $x<z$, the equation gives $S(z) < z + M(z-N(z)) \leqq Mz$. Hence it holds for all x, and $\overline{\lim}\ S(x)/x$ is finite.

If the upper density is zero, choose an x_0 and a K such that $S(x) > Kx$ for $x \geqq x_0$ (choose $K=0$ if necessary). Then the equation gives $S(x) > x + (x - x_0 N(x))K$. For the arguments of all the S's on the right side of the equation add up to $x-n$, and discarding those S's whose arguments are less than x_0 can diminish this sum by at most $(x_0-1)N(x)$, leaving $x - x_0 N(x)$ as a lower bound for the sum of the arguments of the rest, for each of which the function is at least K times the argument. Since the upper density is zero, an x_1 can be found so large that for $x \geqq x_1$, $x_0 N(x) K < x/2$. This gives $S(x) > (K + \frac{1}{2})x$ for $x \geqq x_1$. Hence any lower bound for $\underline{\lim}\ S(x)/x$ can be increased by $\frac{1}{2}$, and $\underline{\lim}\ S(x)/x$ must be infinite.

Also solved by P. J. Owens.

Sequence Related to Laurent Series

4598 [1954, 476]. *Proposed by I. J. Schoenberg, University of Pennsylvania*

Let the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \qquad (r < |z| < r^{-1}),$$

converge in an open ring containing the circle $|z|=1$. Show that a sequence $\{b_n\}$ exists satisfying the conditions

(i) $|b_n| < K$, for some K, for all $n = 0, \pm 1, \pm 2, \cdots$,

(ii) $\sum_{n=-\infty}^{\infty} a_n b_{m-n} = 0$ for all m,

(iii) $b_n \neq 0$ for some n,

if and only if $f(z)$ vanishes somewhere on the circle $|z|=1$.

Solution by Bruce Kellogg, University of Chicago. If a sequence satisfying the required conditions exists for the Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, convergent in a neighborhood of $|z|=1$, and if $g(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ is convergent in a neighborhood of $|z|=1$, then the sequence $\{b_n\}$ also satisfies the required conditions for the Laurent series $f(z) \cdot g(z)$. For if $\sum_n a_n b_{m-n} = 0$ for all m, then

$$\sum_n \left(\sum_p a_p c_{n-p} \right) b_{m-n} = \sum_n \left(\sum_p a_p b_{n-p} \right) c_{m-n} = 0$$

for all m. Hence if $f(z)$ does not vanish on $|z|=1$, $1/f(z)$ may be taken for $g(z)$ and the above result implies that: if a sequence $\{b_n\}$ exists for $f(z)$, it exists for the function $f(z) \cdot (1/f(z)) = 1$, which is a contradiction.

On the other hand, suppose $f(z)$ has a zero, $z=z_0$, $|z_0|=1$. Since there is a sequence $\{b_n\}$ for the function $(z-z_0)$, and

$$f(z) \equiv (z - z_0) \cdot \{f(z)/(z - z_0)\},$$

then there is a sequence $\{b_n\}$ for $f(z)$.

Also solved by the Proposer.

Number of Solutions of a Congruence

4599 [1954, 476]. *Proposed by Leonard Carlitz, Duke University*

Let p be a prime >2, $a \not\equiv 0 \pmod{p}$. Show that the number of solutions of the congruence

$$(x + y + z)^2 \equiv 2axyz \pmod{p}$$

is p^2+1.

Solution by Emma Lehmer, Berkeley, California. First of all we have the trivial solution (0, 0, 0); next, if two of the variables are zero this leads again to (0, 0, 0). If only one variable, say x, is zero, then we have p-1 solutions of the form $(0, y, -y)$, so that altogether we have $3p-2$ solutions in which one or more of the variables is zero.

If none of the variables is zero we can let $y=ux$, $z=vx$, so that the original congruence becomes

$$(1 + u + v)^2 \equiv 2auvx \pmod{p}.$$

Solving for x, y, and z in terms of u and v we have

$$x \equiv \frac{(u+v+1)^2}{2auv}, \qquad y \equiv \frac{(u+v+1)^2}{2av}, \qquad z \equiv \frac{(u+v+1)^2}{2au},$$

with the only conditions on u and v that $u \not\equiv 0$, $v \not\equiv 0$, $u+v \not\equiv -1 \pmod{p}$. Hence there are $p-1$ possible values of u, and for each value of u, except for $u=p-1$, there are $p-2$ corresponding values of v satisfying these conditions. For $u=p-1$ there are $p-1$ such values of v, so that altogether we have $(p-2)^2+(p-1) = p^2-3p+3$ non-zero solutions. Adding this to the $3p-2$ solutions which contain zero we get the required p^2+1 solutions altogether.

Also solved by Bruce Kellogg, D. C. B. Marsh, H. S. Zuckerman, and the Proposer.

Editorial Note. Zuckerman proves the somewhat more general theorem: If p is prime, $a \not\equiv 0 \pmod{p}$, $m \geqq 1$, $n \geqq 1$, then the congruence

$$(x_1 + x_2 + \cdots + x_n)^m \equiv a x_1 x_2 \cdots x_n \pmod{p}$$

has $p^{n-1}+(-1)^{n-1}$ solutions if $n-m$ is relatively prime to $p-1$.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

An Introduction to Deductive Logic. By Hughes Leblanc. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London, 1955. xii+244. $4.75.

This book should be a useful text for an undergraduate course in symbolic logic. The author has managed to combine an elementary presentation of his material with many side glances at some of the manifold problems of modern logic. The book is provided with a good collection of problems.

Chapter one presents an informal introduction to the statement calculus. Truth tables are explained. The chapter contains two final sections introducing the ideas of many valued and modal logics. The second chapter gives a similar informal introduction to quantificational logic. A terminal section provides some information about intuitionistic logic.

Formalization of the sentential and quantificational calculi is carried through in chapter three. The sentential calculus is formalized twice, once with substitutional rules for sentence variables and a finite number of axioms, and once with axiom schemes instead of axioms. Axiom schemes are used for the formalization of quantification theory, but an alternative formalization making use of predicate variables is outlined. It is a useful feature of the book that it familiarizes students with both of these important approaches at an early stage. The chapter concludes with a short discussion of natural deduction.

The fourth chapter introduces the notion of identity. After axioms for equality have been added operators $(\underset{n}{ELx})$, $(\underset{n}{EEx})$, and $(\underset{n}{EMx})$, standing for "there are at least nx," "there are exactly nx," and "there are at most nx," respectively, are defined and shown to have formal properties consistent with their intended interpretation. The remainder of the chapter is devoted to a discussion of classes and relations, and Boolean algebras. The concept of a Boolean algebra is never defined, and the discussion of the relationship between certain logical calculi and Boolean algebras is not as clear or precise as it might be.

The last chapter is of a more advanced nature. The consistency, completeness, and independence of the axioms of the sentential calculus are proved. The quantificational calculus is shown to be consistent and Gödel's proof of the completeness of the quantificational calculus is sketched. A proof that the monadic quantificational calculus is decidable is also included. The book concludes with a few remarks about the identity calculus.

The following misprints were noted: p. 166, line 29, insert n under the first E; p. 197, line 35 change ζ_1 to ζ_i, and next line down change the last ψ in line to ζ; p. 213, line 1, replace ζ by ψ; p. 219, line 12, the subscript under the A should read 2^k, not $2k$.

STEVEN OREY
University of Minnesota

Engineering Cybernetics. By H. S. Tsien. New York, McGraw-Hill Book Co. 1954. xii+289 pages. $6.50.

This valuable book, written for (actual or potential) research engineers and applied mathematicians, deals with the science of control and communication, which N. Wiener has christened cybernetics. The author deliberately gives a purely theoretical treatment of highly practical topics, as is usually done in fluid mechanics. For example, he recognizes but is not afraid of the fact that some engineers will think the book highbrow, while mathematicians will be disappointed if they expect the rigor and elegance that are customary in abstract mathematics (and in some that is called applied). The principal mathematical prerequisites are of course ordinary differential equations and complex integration; variational methods are also used on occasion.

The author has not extended the scope of his work by means of bibliographic material or collections of exercises, but the text itself covers an unusually wide range of topics. This may be indicated by chapter as follows: 1. Introduction. 2. Laplace transform. 3. Transfer function. 4. Feedback servomechanism. 5. Noninteracting controls. 6. Alternating-current and oscillating-control servos. 7. Sampling servos. 8. Time lag. 9. Stationary random inputs. 10. Relay servos. 11. Nonlinear systems. 12. Linear system with variable coefficients. 13. Perturbation theory. 14. Control design with specified criteria (*e.g.*, minimize the time-average of the error-squared). 15. Controls that automatically seek to maximize a criterion of good performance. 16. Filtering of noise (sketches of the Wiener-Kolmogoroff and other theories). 17. Ultrastability and multistability (W. R. Ashby's conception of a system capable of learning in the sense of responding to environmental changes by searching out a stable pattern of behavior for itself). 18. Error control (J. von Neumann's theory of improving the reliability of a system by duplicating or multiplexing its elements). This last study has been extended (to nets of relays) by C. E. Shannon and E. F. Moore in some as yet unpublished work.

In Chapter 9 it appears that the author should have had the advice of someone familiar with statistics and probability. The distribution (9.49), often called exponential, is never called Poisson's distribution. Above (9.7), the terms variance and mean deviation should not be used for the standard deviation σ. The most serious lapse noted concerns the concept of ergodicity or metrical transitivity; it not only goes unnamed, but is described as if it were identical with stationarity, or were a consequence thereof.

In general, however, the book appears to be adequately clear and accurate. Also, the author has done his duty by a new and growing subject, by including much material (from recent research papers) which has not previously appeared in book form.

E. L. KAPLAN
Bell Telephone Laboratories, Inc.

Existence Theorems for Ordinary Differential Equations. By F. J. Murray and K. S. Miller. New York University Press, 1955. x+153 pages. $5.00.

This book is an unhurried development of classical existence and uniqueness theorems for ordinary differential equations.

The leisurely pace of the book is set in the first two chapters of 32 pages devoted to six standard existence theorems. In Chapter 3 hypotheses are added to theorems of the first two chapters to secure sufficient conditions for unique solutions. Chapter 4 treats the Picard iterants. Properties of solutions, such as differentiability of solutions with respect to parameters, are discussed in Chapter 5. The final, and longest, chapter is devoted to linear differential equations.

In most instances the theorems are stated for real functions and, except for the linear equations, the theorems are usually for solutions in the small. Riemann integration is used throughout. The book gives the impression of being a good set of lectures rather than a polished exhibition prepared especially for publication. Thus the notation is modern but not condensed and the theorems, proofs, and examples are thoroughly discussed.

Students possessing some background in function theory will find this book a readable and reliable account of an important topic in analysis.

W. R. Utz
University of Missouri

General Topology. By J. L. Kelley, New York, D. Van Nostrand Co., Inc., 1955. xiv+298 pages. $8.75.

This book is intended as "a systematic exposition of general topology which has proven useful in several branches of mathematics" especially analysis. As such it is thoroughly successful and should provide an excellent reference volume for those mathematicians who are interested in this subject as a *Gebrauchtopologie.*

The book begins with Chapter 0 on set theory, relations, orderings and the like, culminating in a theorem which proves eight statements of the extremum law = Hausdorff maximal principle = Zorn's lemma *etc.* to be equivalent. Chapter 1 introduces the notion of a topology and explores some of its implications; Chapter 2 deals with Moore-Smith convergence. Chapter 3 covers product and quotient spaces, while Chapter 4 on embedding and metrization proves the standard metrization theorems. The latter chapter also investigates some of the properties of pseudometric spaces. In Chapter 5 on compact spaces the productive and divisible properties of compact spaces are discussed, the usual compactification theorems are presented, and finally paracompactness is briefly presented. Chapter 6 concerns uniform spaces which are presented *à la* Bourbaki. The pseudo-metrization theorem for uniform spaces is proved as well as the uniqueness of the uniform topology on a compact space. A form of the Baire category theorem for pseudo-metric spaces closes the chapter. Chapter 7 on function spaces lies somewhere between general topology and analysis. It in-

vestigates various topologies that may be introduced in spaces of functions and ends with a proof of the Ascoli theorem. This is presented first in the usual form for equicontinuous families and then in the author's form for evenly continuous families. Finally there is an extensive appendix on elementary set theory, which develops this theory axiomatically.

The problems at the end of each chapter contain a large part of the theory. Some are simple exercises, examples and counter examples which form, as the author indicates, a necessary adjunct for the understanding of the theory. Others are major theorems and theories; for example the Tietze extension theorem and the Stone-Weierstrass theorem occur as exercises. Suggestions for the proof of the more difficult of such theorems is given by means of hints or a suggested sequence of steps (lemmas) so that the mature student should be able to carry out the details with only minor difficulties.

This volume has considerable value as a reference book. On the other hand, the reviewer feels that it has only limited utility as a text. For a course in topology *per se* it is unsuitable since it does not prove many of the purely topological results. As an example may be cited that the topological characterizations of the arc and simple closed curve do not occur either in the text or as exercises. Its usefulness as a text would come in a course on topology for analysts. Such a course could cover the major part of the book in one semester if a knowledge of Chapter 0 were presupposed.

The book has an attractive format and the reviewer found but few typographical errors. The style is quite readable and, if this may be said of mathematical writing, sprightly.

J. D. Baum
Oberlin College

OBITUARY

JULIAN LOWELL COOLIDGE

In Memoriam

I

There have been Coolidges in America ever since John Coolidge came to Massachusetts Bay with the first settlers and made himself a home in Watertown. His descendants have spread and formed their own families, of which members have made a name in public life, in the law, in the arts and in the sciences. Many of them have remained faithful to New England, its institutions and traditions. Here also Julian Lowell Coolidge grew up and worked, not only a New Englander, but in particular a Harvard man. With Harvard he was

associated through family associations and through intimate personal connections, which began with his undergraduate years in the nineties and ended only with his death.

He was born in Brookline, Massachusetts, September 28, 1873, the son of Joseph Randolph Coolidge and his wife Julia Gardner Coolidge. The father, who was a member of the Massachusetts bar, provided his children with a good education which included travel. Julian, after a year at Exeter, entered Harvard College in 1892, where he received his B.A. in 1895, *summa cum laude*. Although at that time he did not yet think of a career in mathematics, there was already enough activity at Harvard to stimulate a budding scientist of exact leanings, and the young student knew how to profit by it.

Coolidge himself has later described the scientific climate of the Harvard of his early days: "The great museum bore the name of the elder Agassiz, and was ably managed by the younger. Botany brought to mind the names of Goodale and Farlow; psychology, James and Münsterberg. In chemistry a great figure was passing in Cooke, in geology the astounding Shaler had a wide influence. In mathematics three men had borne the heat and burden of the day since the death of Benjamin Peirce, namely his son, James Mills Peirce (A.B.1853), William Elwood Byerly (A.B.1871) and Benjamin Osgood Peirce (A.B.1876), a distant relative of the other Peirces." [15] Of those men, Benjamin O. Peirce became a good friend.

Mathematics had recently been strengthened by the appointment of two younger men, William Fogg Osgood of the class of 1886 and Maxime Bôcher, class of 1888. Both had studied at Göttingen under Felix Klein, both had written dissertations which still are read, both set their permanent imprint on the Harvard mathematics division as scientists as well as teachers. However, in Coolidge's undergraduate days Harvard had not yet advanced to a school where students worked for a higher degree in mathematics, and so Coolidge, intent on making teaching his career, set out for Oxford, where at Balliol he finished with a B.Sc., the first degree in natural science awarded by the hitherto severely humanistic university. After his return he settled as a schoolteacher, and from 1897–99 taught at the school in Groton, Massachusetts, at that time under the leadership of its founder, the Reverend Endicott Peabody. Among his pupils was Franklin D. Roosevelt, with whom Coolidge always maintained a friendly relationship.

Archibald Cary Coolidge, Julian's elder brother, later founder and head of the Widener Library, had at that time entered the Harvard history department after a diplomatic career abroad. Under his influence Julian accepted a position as instructor in the mathematics department—initial salary $250 a year. From this period date his first publications, which already show the trend of his interests: one on a geometric representation of imaginary elements in the plane, the other on the construction of the intersection of two conics with a common focus with the aid of compass and straight edge alone, if at least three points or tangents of each conic are given [27]. This construction has a neat application

to the problem of Apollonius, to construct a circle tangent to three given circles. There exists another paper of these years concerning the classification of quadric surfaces in hyperbolic three space [3]. The classification is based on the intersection of the surfaces with the absolute, and Coolidge improves on Clebsch by sharply discriminating between real and imaginary elements, and between the regions inside and outside the absolute.

In 1902 Coolidge became a member of the Faculty and was granted a two year leave of absence to study abroad. He was a married man now, and accompanied by Mrs. Coolidge, he first went to Paris, and then to Greifswald. Here he studied under Kowalewski and especially under Study. Afterwards he travelled to Turin, where he heard D'Ovidio and in particular Corrado Segre; then returned to continue his studies with Study, who had moved from Greifswald to Bonn. There, in 1904, he received the doctor's degree with a thesis on non-euclidean line geometry, a field in which both Segre and Study had done outstanding work [5].

These years, 1902 to 1904, were decisive for Coolidge's mathematical career. With both Segre and Study he formed strong scientific and personal bonds, to which he repeatedly refers in later work. The thesis was written in almost daily contact with Study, who had just written his *Geometrie der Dynamen* —essentially a highly original line geometry in euclidean space—and whose methods Coolidge learned to apply. The thesis is still an interesting bit of research; those who like to study its ideas can also find them in Coolidge's first book, that on Non-euclidean Geometry, in a section that has always struck us as one of the most interesting parts of this book. We also find here the essentials of another paper by Coolidge dating from the time of his study with Study and Segre. This paper, placed by Segre in the Atti of the Turin Academy, dealt with isotropic congruences of lines in elliptic space—congruences of which the focal surface is a developable circumscribed to the absolute [6]. Here again we meet a subject which always fascinated Coolidge: the geometrical interpretation of complex numbers and their functions.

Back home, Coolidge continued teaching at Harvard, where he became an assistant professor in 1908. In those days he began to work on the first of his books and approached the Clarendon Press in Oxford to find out whether it was interested. The first result of the agreement between author and publishing house was the *Non-euclidean Geometry* of 1909. This was the beginning of an association which lasted for fifty years: all eight of Coolidge's books have been published by the Clarendon Press. These books have become well known in professional circles for their clear exposition, the excellence of the material, the facility of their style and their personal touch. They also brought the author's own contributions, organized in the framework of the general text.

The papers of this first period after the return from Bonn belong in the main to the Segre-Study school, with its generous touch of Lie. In 1909 appears Coolidge's first article on the theory of probability, in 1915 his first on the algebraic theory of curves [11, 17]. Life at Harvard continued at an even pace in a

growing department; in 1912 he attended the International Mathematical Congress at Cambridge, England. In 1916 he published his second book, the *Treatise on the Circle and the Sphere*. In 1918 he became a full professor.

The world war interrupted for a while the work at Cambridge. Coolidge served as a major in the United States army, and from 1918–19 was an A.E.F. liaison officer with the French general staff, a function in which his fluent command of French was of great help. In 1919 courses were organized at the Sorbonne for American officers and enlisted men who still had to stay in France; Coolidge was placed in command of the 1800 men involved and spent much time and effort in making the best of an unusual and arduous assignment in which he had, it has been said, the duties of a dean without the sanctions usually at the disposal of such a dignitary. However, he seemed to have found the teaching and administrating assignment quite rewarding. It also brought him the *Légion d'honneur*, as *Chevalier* in 1919, as *Officer* in 1936.

After the war Coolidge resumed his duties on the Harvard faculty. In 1924 he published the *Geometry of the Complex Domain*, still a tribute to his Bonn and Turin days. He had lectured on this subject at the Sorbonne in 1919, he again took it up in a lecture given at Rome in 1926 on a visit from Paris, where he stayed during 1927 as an exchange professor at the Sorbonne [**42**]. At that time his *Introduction to Mathematical Probability* (1925) had already appeared. Of all Coolidge's books this one is probably the best known; Coolidge found this out not only because it received a German translation in 1927, but also through word from his publishers that it had a vogue at Monte Carlo. In 1927 he became chairman of the mathematics department, where he organized the main features of that department's tutorial system. In 1929, after President Lowell had inaugurated his so-called House Plan for the reorganization of the college into smaller residential units, Coolidge accepted the position of Master of Lowell House. It was a delicate assignment in a new university venture of considerable consequence [**59**], which gave Coolidge the opportunity to build an atmosphere of friendly social life combined with serious scholarship. The task was congenial to him and he stayed at Lowell House until 1940, when he retired both as professor and as master. During the period of Lowell House Coolidge finished his *Algebraic Curves* (1931) and his *History of Geometrical Methods* (1940). The first was a token of his indebtedness to the geometers of the Italian school—it is dedicated "*Ai geometri italiani, morti, viventi*"—the latter opens the series of his books on the history of mathematics, a field in which we see his interest growing after 1924, the year of first paper on the subject. The other two historical books were published when Coolidge was already retired; the *Conic Sections and Quadric Surfaces* dates from 1943, the *Mathematics of Great Amateurs* from 1949. Retirement, however, did not prevent him from offering his services to the university during the war emergency, and so we see him, during 1942–43, return to his old love, the classroom, where he taught the calculus to enlisted men. He was productive well-nigh till the end, his last paper, on the length of curves

[71], appeared in 1953, when he was in his eightieth year. He died March 5, 1954. He is survived by his wife, Mrs. Theresa Reynolds Coolidge, five daughters and two sons.

Coolidge was deeply interested in both the Mathematical Association of America and the American Mathematical Society. He was vice-president of the Association in 1924 and president in 1925, member of the Society's Council during 1911–13 and 1924–25, vice-president in 1918 and a trustee in 1929–30. During his year of service as President of the Association Coolidge suggested the establishment of the Chauvenet Prize for noteworthy expository writing in mathematics. He himself donated $100 to the Association to be used for the first award of the Chauvenet Prize. His most important contribution to the Society was his chairmanship of the Endowment Fund drive. The campaign to place the finances of the Society on a firm foundation was inaugurated in 1922 and lasted till 1926; it was not only projected as a fund-raising device, but also as an educational venture, to place before the public the basic character of mathematics and mathematical research in our civilization. There were newspaper and magazine articles and the inauguration of the Josiah Willard Gibbs Lectureship. By 1926 the chairman could feel satisfaction: the financial condition of the Society was improved by about $70,000 in gifts, sustaining memberships and a subvention from the National Academy of Sciences. The main burden of the drive, which often consisted in day by day soliciting, was carried by Coolidge, assisted by Professors Veblen and Dresden.

Coolidge's wit was proverbial, it leavened his teaching, his conversation and, we are glad to say, his written work, so that readers can still enjoy it though the writer has passed. It is strong in his book reviews, where the personal touch gives the feeling of direct contact between reader and author, but also tickles us in his textbooks, as in his comment on the unsatisfactory character of Bayes' theorem in probability: "Steyning tuk him for the reason the theif tuk the hot stove—bekaze there was nothing else that season." Suspecting (and by the way, rightly so), that his collection of "mathematical amateurs" might be judged rather arbitrary, he announced: "If consistency is a vice of small minds, it is a vice I have successfully avoided." Even when engaged in a stern mathematical reasoning he liked to jump off the high horse and allowed his readers or his audience to relax through a colloquialism or a personal remark.

Teaching was Coolidge's first professional love and he continued to cherish it throughout his life, building up a reputation as an excellent instructor with a plastic way of reasoning. He liked to do strenuous things, as a student at Harvard he gained a gold medal for his record-breaking mile, 4 min. and 30 4/5 sec.; at Balliol he was on the crew. He indulged in hiking and swimming, and loved to sail off the Maine coast, where he had a summer home. Interested in the conservation of our national resources, he forested himself a large tract of woodland. He was also an amateur astronomer of considerable vigor; later in life he presented the 6-inch telescope at his house in Cambridge to the American

Association of Variable Star Observers, who have put it to new duty in Minneapolis.

We have already mentioned his relation to the Clarendon Press. An extract from a letter by Mr. A. M. Wood of this house may properly finish this section of our paper:

"He first got in touch with the Press in 1904, at a time when we did very little mathematical publishing. He continued to publish with us throughout his working life and our relations with him were uniformly happy. He used to tell us of his plans for writing and, looking through past correspondence, I am impressed by the way in which he saw each book whole before he started. He would tell us that his next book would be on a certain topic; he would outline its scope and its relationship with other books in the field and would tell us how many years it would take him to write it. His estimates of this time were very accurate."

Coolidge was lucky with his publishers: few mathematicians can boast of so many beautifully printed books.

II

The Bonn thesis [5] is a study of lines in an elliptic three-space S_3. Such lines have a polar with respect to the absolute, line and polar together form what Study has called a *cross*. The figure is equivalent to that of two totally perpendicular planes at a point in euclidean four-space. It is also equivalent to a pair of points in an elliptic plane, or to its dual, a pair of lines in such a plane. Two real lines in S_3 which meet the same pair of left (right) generators of the absolute are called left (right) *paratactic*, a term introduced by Study for what often is called *Clifford parallel*. Such left (right) paratactic lines have an infinite number of common perpendiculars and are at constant distance. Through a point of the S_3 one right and one left paratactic can be drawn to the lines of a given cross. A cross can therefore be represented by two lines through (1, 0, 0, 0), the left paratactic has now coordinates ${}_1X_i$, $i=1, 2, 3$, different from zero and proportional to the coordinates of intersection with the plane $X_3=0$. There are also 3 coordinates ${}_2X_i$ for the right paractactic. The planes ${}_1X$ and ${}_2X$ are the representative planes. There is a dual representation ${}_1U$, ${}_2U$. Coolidge now introduces for the coordinates of the cross not the X themselves, but the "dual" combinations

$$ {}_1X_i\epsilon_1 + {}_2X_i\epsilon_2 = X_i, \qquad {}_1U_i\epsilon_1 + {}_2U_i\epsilon_2 = U_i, $$

where

$$ \epsilon_1^2 = \epsilon_1, \qquad \epsilon_2^2 = \epsilon_2 > \epsilon_1\epsilon_2 = \epsilon_2\epsilon_1 = 0. $$

The properties of crosses can be very elegantly expressed in terms of these dual coordinates; *e.g.*, the scalar product $U \cdot X = 0$ means orthogonality. The group of this dual projective geometry G_{16} is the group of linear transformation of the X_i with dual coefficients.

The object of the dissertation is to study certain systems of crosses. Those which are selected are mapped on the representative planes by two projective point ranges (a *chain*), by a point in one plane and a line in the other (a *strip*) and a collinear relation between the planes (*homographic congruence*). In the case of a chain the *chain surface* in S_3, belonging to the general chain in the form

$$_2X_1 = a_1\,{}_1X_1, \qquad {}_2X_2 = a_2\,{}_1X_2, \qquad {}_2X_3 = {}_1X_3 = 0, \qquad a_1^2 = a_2,$$

is of fourth order and class and its own polar, with two mutually absolute polar double lines and two absolute generators of each set and given by

$$(a_1 - a_2)(x_0^2 + x_3^2)x_1x_2 + (a_1 + a_2)(x_1^2 + x_2^2)x_0x_3 = 0$$

in projective coordinates.

The strip surface is the well-known Clifford surface of zero curvature. The general homographic congruence is of the third order and class, is its own absolute polar, contains all generators of the absolute and may be constructed in 15 ways as partial intersection of two of six tetrahedral complexes.

This dissertation has many traits which are constantly returning, in some form or another, throughout Coolidge's later work; especially his interest in non-euclidean geometry, in line geometry and his preoccupation with the geometry of complex numbers, as well as his solid technical skill in handling the analytical apparatus of a particular geometry. Most of his later results he embodied in his books, and we can therefore do fair justice to them by following the trend of these books.

The *Non-euclidean Geometry* [10] was one of the first books on this subject written in English. The author had several foreign examples to draw from, especially the works by Killing and Liebmann, but he preferred to follow his own approach. He chose an axiomatic introduction, selecting his axioms so that at one point he could make his choice of three avenues: one leading to euclidean, and the two others to the two non-euclidean geometries. After the synthetic buildup he develops the trigonometry and the analytic formulae with the extension to space and congruent transformations. The author's love, as he tells us, is in fruits rather than in roots, and so we get a full treatment of conics and quadrics, where he can utilize his own work [3]. Areas and volumes follow. Of permanent value is the algebraic and differential geometry of line and line crosses where we find the subject of the thesis anew. Here we also meet Study's representation of all real rays of elliptic space on pairs of real points of two euclidean spheres, which establishes a relation between analytic functions of a complex variable and the isotropic congruences in elliptic space [6]. The importance of this representation has since been brought out again by Blaschke (Annali di Matem., vol. 28, 1940, p. 205), now with the use of quaternions.

Interest in line geometry easily leads to interest in sphere geometry by means of the transition from Plücker coordinates into pentaspherical coordinates. They play an important role in Coolidge's *Treatise on the Circle and Sphere* [19].

However, circles and spheres have an importance beyond that indicated by Klein and Lie transformations, as already Euclid has shown. Coolidge tried to bring some organization into the colossal mass of literature dealing with circle and sphere, and so we have a book in which we can find the circles of Apollonius, Feuerbach, Lemoine and Brocard as well as the chains of DeLongchamps (these and other sets of circles were favorites with Coolidge, see [12] and [18]), the cyclids of Darboux and the results of Laguerre and Lie. Among Coolidge's own contributions we find his proof that equilong transformations of the plane depend on an analytic function of one complex variable (or the conjugate complex) [7], his study of metrical relations in Lie transformations by comparing distances on lines with angles of spheres, keeping in mind that the contact transformations which carry lines into lines form a G_6, while those that leave the angles of spheres intact form a G_{10} [13]. Restriction of this group leads to a close analogy between non-euclidean line geometry and the euclidean of the so-called circle cross. Such a circle cross, corresponding to Study's line cross, is composed of a pair of circles so related that every sphere through one is orthogonal to the other. There are also *paratactic* circles, which are cut twice orthogonally by at least one third circle, and hence cut orthogonally twice by ∞^1 circles. This theory leads to the so-called pseudo-cylindroid, which is a one-parameter family of circle crosses dependent linearly upon those of two fixed crosses and is a surface of the eighth order [14]. In this book we find also Coolidge's theory of congruences, as well as of complexes of circles in space [16]. It is impossible here to do full justice to this highly original treatise so full of pleasant theorems; it has had considerable influence on the later development of conformal algebraic and differential geometry. In later life Coolidge also tried his hand at congruences of conics in space [54], where the methods used for circles must be changed; he succeeded by applying Darboux' method of the trihedron moving with the conic with respect to a fixed trihedron.

In the *Geometry of the Complex Domain* [35] Coolidge returns, again in a different way, to the ideas of Segre and Study. The central thought is the geometrical representation of points with complex coordinates in the binary and ternary domain and the investigation of their configurations in their own right, and not, as is customary, as direct generalizations of configurations in real space. The central concept is the chain, which is defined in the complex binary domain as collection of points for which a) the cross ratios of any four are real, and b) for which there exists one point having with any three points of the system any given real cross ratio different from 0, 1, ∞. A point is defined as an object in one-to-one correspondence with a pair of homogeneous complex coordinate values (x_1, x_2) not both zero. In the Gaussian plane the chain appears as a circle or line. Any collineation and anti-collineation carries a chain into a a chain—we see how close we are again to non-euclidean and to circle geometry. Involutory anti-collineations have fixed points defined by an equation of the form $(ax)(\bar{a}\bar{x})=0$ ($\bar{x}$ conjugate complex to x) and thus we arrive at Hermitian forms, and later to forms $(ax)^n(\bar{a}\bar{x})^n=0$. These Hermitian forms and their geom-

etry were subject of special papers by Coolidge [23, 26]. Much of the text is taken up by the theory of chain congruences and their relation to Hermitian metrics. Differential properties are included as well. The book ends with complex space of three dimensions and an axiomatic treatment of complex geometry based on Von Staudt. It was in some respects a forerunner of modern Hermitian geometry, just as the previous book anticipated modern conformal geometry.

Where the three previous books form a kind of loose trilogy, the fourth geometrical treatise has a position of its own. Coolidge's first paper on algebraic curves [17] dealt with his attempt to attach a genuinely geometrical meaning to the order n of a plane algebraic curve $a^n x = 0$ with at least one real continuous branch. He found it by defining the order as the number greater by unity than the number of successive polar systems necessary to determine the curve, and on this based definitions of class and deficiency. Somewhat later he proved that the only algebraic identities involving any combination of real or total singularities, which are valid for all real algebraic plane curves, are those deducible from the Plücker and Klein equations [20]. There are 16 characteristics denoting their irregularities, of which seven are "total," that is, defined without taking the distinction between real and imaginary into account. The theory of algebraic curves continued to interest him [31, 34, 41, 44] and eventually brought him to write his book on plane algebraic curves [46]. Here, as in the case of his first book, he entered a field in which there was a need for a modern English book, though several good books in other languages existed.

The method of the book is algebraic, though no reference is made to algebraic number theory. In the exposition stress is laid on the concept of correspondence. There is much on real circuits of curves and on the reduction of singularities, and on systems of points on a curve. Linear and non linear systems of curves are studied, the latter in the form indicated in [44]; and a good deal is said about Cremona transformations. The emphasis is on the general curve rather than on curves of the third and fourth degree, though they are not neglected. The whole treatment is strongly influenced by the Italian school of Bertini, Segre, Castelnuovo, Enriques and Severi, all inspired by the genius of Max Noether.

At present, with so many books on probability in the English language, it is a little strange to realize that Coolidge's work on this subject [37] appeared at a time that there existed hardly any modern English treatise in this field. Coolidge here followed the example he set in two of his previous books, so that his text helped to open a new period of interest in probability, at any rate in the English speaking countries, an interest which did no longer center on the classical theory of games, but also allowed ample room for error and correlation theory and statistics in general. However, Coolidge's first paper on probability [11] was still on the theory of games; it elaborated on variation of the problem of the ruin of the player. It contains a moral lesson:

"The average gambler will say 'the player who stakes his whole fortune on

a single play is a fool, and the science of mathematics can not prove him to be otherwise.' The reply is obvious: 'The science of mathematics never attempts the impossible, it merely shows that other players are greater fools'."

Some papers of a more statistical nature followed [**24, 27, 32**], which eventually found their niche in the book; however, it was not only the attention paid to statistics which added interest to the book. One of its features was its break with the classical definitions of probability as a measure of ignorance, and its introduction of the so-called frequency definition. This definition took probability out of the field of abstract mathematics and played it into that of the natural sciences. We shall not evaluate its merits and demerits here, suffice to say that this approach to probability had also strongly been argued in Germany by Von Mises and others of his school. The favorable review of the book in the *Jahresbericht der Deutschen Mathematiker Vereinigung* stressed this aspect; and shortly afterwards a German edition appeared [**37a**]. The book makes entertaining reading and has a particular Coolidge touch, with its already quoted crack from Kipling and the following formulation of a famous 18th century problem:

"If all the inhabitants of Chicago should meet together in one place and get extremely drunk, and then try to go home by guess-work, the chances that at least one would get back to his own bed are almost two out of three."

Coolidge would not for the world have substituted Boston for Chicago, or Cambridge for Boston.

The three books on the history of mathematics are full of thorough information, the result of the mature thinking of a scientist about the way his own field of learning came into being. The *History of Geometrical Methods* [**55**], written in independent emulation of Chasles' century old *Aperçu Historique*, is a broad survey of almost all fields of geometrical endeavor from the Greeks to the present. In his mastery of so many different branches, his understanding of the relations between them and the directness of his approach, Coolidge endowed his text with something of the spirit of Felix Klein. The book will remain an excellent introduction to all these fields of geometry which were predominant in the nineteenth century, while keeping a window open to the present. The *History of Conic Sections and Quadric Surfaces* [**61**] is not only a report on the many authors who, from Menaechmos on, built up our present theory of conics and quadrics; it is a critical appraisal of their work as well, and a good and quite original introduction into the theory. The *Mathematics of Great Amateurs* [**64**] is a collection of essays on several men of the past who contributed to mathematics but were essentially prominent in other fields, such as Plato, Omar Khayyam and Diderot. For some reason Napier and L'Hospital are also included. The merit of such a book lies in the fact that here we find a modern professional opinion on the mathematical works of these authors of a by-gone age by one who has taken the trouble to read their works. Particularly valuable to those who are interested in geometry and art are the chapters on Pietro dei Franceschi,

Leonardo Da Vinci and Albrecht Dürer. Some more on this subject can be found in Coolidge's review of Ivins' book [78].

The many smaller papers on the history of mathematics cover different territory. Some deal with the growth of American mathematics [36, 38, 45, 52, 56] some with persons [29, 38, 40, 48, 50, 58, 60, 68], some with whole fields [43, 49, 51, 63] and several, especially those of his later years, with special topics on which he was able to throw some historic light [53, 65, 66, 67, 69, 70, 71]. The latter are particularly useful to instructors and college textbook writers who like to obtain a better grasp of the historical foundation of their topics.

More than a passing word should be said about Coolidge's book reviews. Let us not forget, there are reviews and reviews. Many do not figure in the creative output of a person, they are written for little else than duty, the desire to own a free copy, or to see one's name in print. But there are reviews that must count as creative writing. Reviews are a common form of literary criticism, one of our most discussed philosophical treatises is in the form of a book review, Leibniz first announced the integral calculus in a book review, and Asa Gray's review of Darwin's *Origins of Species* made biological history. When Coolidge reviewed a book which caught his fancy, he gave it his whole attention, matched his wits against those of the author, spared no praise nor spared the rod, and by the time he was through the review had become a veritable essay. Several of these reviews are even now instructive and can be evaluated as regular papers comparable to [33] or [62]; they contain some of that science-between-the-lines which is usually reserved for the classroom or for private conversation, and is responsible for half the fun of teaching and study. Our bibliography contains some of these valuable book reviews.

It is a curious fact that with all his vigorous scientific activity Coolidge always seems to have labored under the thought that his type of mathematics was somehow passé, that he was in a certain sense an epigone. With him it was not a reaction due to old age, a phenomenon common enough among scientists. We find the thought expressed in the very first sentence of his very first book, written when Coolidge was still fresh from Bonn: "The heroic age of non-euclidean geometry has passed." Later we see him writing about the "rise and fall" of projective geometry [49]. There are more such places in his writings. It is true that certain methods and certain aspects of geometry with which Coolidge was particularly familiar had passed their prime. But Coolidge, in his diffidence, may not have seen as clearly as others how his own work contributed to that rejuvenation through which all science must pass in order to remain alive. Coolidge, thinking in 1909, that the hey-day of non-euclidean geometry was over, did not foresee the great flowering of non-euclidean geometries which came in the wake of Einstein's theory of relativity. And we have already seen that some of his books can even now be considered as pioneer ventures in fields that are very much alive. Significant mathematics is never antiquated, even if there are periods in which certain fields lie dormant.

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D. J. STRUIK
Massachusetts Institute of Technology

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

BELL TELEPHONE LABORATORIES FELLOWSHIPS

The Bell Telephone Laboratories has announced the establishment of a fellowship program through which it will grant funds for students doing graduate study in electrical communications.

To be known as the Bell Telephone Laboratories Fellowships, the awards are for study of one or two years leading to a doctorate. Each fellowship carries a grant of $2,000 to the fellow and an additional $2,000 to cover tuition, fees, and other costs to the institution at which he chooses to study.

Recipients of the fellowships will not be required to limit their study to

electrical engineering, although the field of study and research must have a bearing on electrical communications. They may, for example, pursue various branches of mathematics, physics, chemistry, engineering mechanics, and mechanical engineering. Fellows may make their own choice of an academic institution within the United States.

EDUCATIONAL TESTING SERVICE PSYCHOMETRIC FELLOWSHIPS

The Educational Testing Service is offering for 1956–1957 its ninth series of research fellowships in psychometrics leading to the Ph.D. degree at Princeton University. Open to men who are acceptable to the Graduate School of the University, the two fellowships each carry a stipend of $2,500 a year and are normally renewable. Fellows will be engaged in part-time research in the general area of psychological measurement at the offices of the Educational Testing Service and will, in addition, carry a normal program of studies in the Graduate School.

Suitable undergraduate preparation may consist either of a major in psychology with supporting work in mathematics, or a major in mathematics together with some work in psychology. However, in choosing fellows, primary emphasis is given to superior scholastic attainment and demonstrated research ability rather than to specific course preparation.

The closing date for completing applications is January 12, 1956. Information and application blanks may be obtained from: Director of Psychometric Fellowship Program, Educational Testing Service, 20 Nassau Street, Princeton, New Jersey.

PERSONAL ITEMS

Professor J. R. Mayor of the University of Wisconsin has been appointed Director of the AAAS Science Teaching Improvement Program.

Georgia Institute of Technology announces the following: Associate Professor M. B. Sledd has been promoted to a professorship; Assistant Professor J. A. Nohel has been promoted to an associate professorship; Dr. W. R. Smythe, Jr., of Duke University has been appointed to an assistant professorship.

Michigan State University reports: Professor J. S. Frame, head of the Department of Mathematics, is temporarily serving as Acting Head of a new Department of Statistics, recently formed at the University; Professor Joseph Meixner of the Technische Hochschule, Aachen, Germany, has been appointed Visiting Professor for the year 1955–1956; Dr. M. L. Tomber of Amherst College and Assistant Professor R. L. Blair of the University of California at Davis have been appointed to assistant professorships; Mr. S. K. Berberian, Mr. C. C. Faith, and Mrs. Delia Koo have been appointed to instructorships.

United States Naval Postgraduate School anounces: Assistant Professor E. J. Stewart of California State Polytechnic College has been appointed to an associate professorship; Associate Professor B. J. Lockhart has been promoted to a professorship; Professor C. H. Rawlins has retired with the title of Professor Emeritus.

University of Connecticut reports the following: Dr. Felix Haas of Princeton University and Dr. R. P. Gosselin of Youngstown College have been appointed to assistant professorships; Dr. J. A. Schatz of Lehigh University and Dr. Violet B. Haas of Immaculata College have been appointed to instructorships; Dr. Lida K. Barrett has been appointed to an instructorship at the Waterbury Branch.

At the University of Nebraska: Assistant Professors Edwin Halfar and L. K. Jackson have been promoted to associate professorships; Dr. Arne Magnus has been promoted to an assistant professorship; Mr. D. W. Miller and Dr. E. J. Schweppe have been appointed to instructorships.

University of New Hampshire announces the following: Mr. R. W. Sloan of the University of Illinois has been appointed to an assistant professorship; Dr. S. L. Ross of Boston University has been appointed to an instructorship; Assistant Professors J. B. Crabtree and H. G. Rice have been promoted to associate professorships.

University of Washington reports: Dr. Anne C. Davis of the University of California, Dr. Joel Franklin of New York University, and Dr. R. S. Pierce of Harvard University have been appointed to assistant professorships; Mr. Alfred Descloux of the University of North Carolina has been appointed to an instructorship; Dr. H. C. Wang of the Institute for Advanced Study has been appointed Visiting Lecturer; Dr. A. E. Livingston and Dr. R. F. Tate have been promoted to assistant professorships; Professor Edwin Hewitt is on leave of absence for the academic year 1955–1956 on a Guggenheim Fellowship at the Institute for Advanced Study; Associate Professor D. G. Chapman has returned from his leave of absence at Oxford University; Professor R. A. Beaumont, who was a Ford Fellow at the Institute for Advanced Study, has returned from his leave of absence.

Dr. S. S. Abhyankar, previously a teaching fellow at Harvard University, is now an associate in mathematics at Columbia University.

Dr. C. M. Ablow, formerly a research engineer for Boeing Airplane Company, Seattle, Washington, has a position as a senior research mathematician for Stanford Research Institute, Menlo Park, California.

Mr. Walter Abramowitz, previously a physicist for Solid State Research Institute, New York City, now has a position as an analytical engineer for Pratt and Whitney Aircraft, East Hartford, Connecticut.

Mr. W. R. Allen, formerly an associate at Forrestal Research Center, Princeton University, is now a member of the technical staff of the David Sarnoff Research Center, Radio Corporation of America Laboratories, Princeton, New Jersey.

Associate Professor A. G. Anderson, chairman of the Department of Mathematics of Duquesne University, has accepted a position as a mathematician at the Research Center of Jones and Laughlin Steel Corporation, Pittsburgh, Pennsylvania.

Mr. W. P. Anderson, formerly an instructor at the University of Minnesota,

has a position as a mathematician at the Rand Corporation, Santa Monica, California.

Mr. R. R. Archer, previously a student at Massachusetts Institute of Technology, has been appointed to an instructorship at the Institute.

Dr. W. G. Bade of Yale University has been appointed to an assistant professorship at the University of California.

Professor L. C. Bagby of Lawrence Institute of Technology has accepted a position as staff industrial engineer with Burns and Roe, Detroit, Michigan.

Assistant Professor R. W. Bagley of the University of Kentucky has a position as a senior aerophysics engineer at Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas.

Associate Professor G. A. Baker of the University of California at Davis has been promoted to the position of Professor of Mathematics and Statistician in the Experiment Station.

Mr. Sheldon Balk, formerly a graduate student at Arizona State College, has a position as an equipment engineer for the Douglas Aircraft Company, Tucson, Arizona.

Mr. C. W. Barnett, previously a mathematician in the Mathematical Analysis Department of Lockheed Aircraft Corporation, Marietta, Georgia, is now a research programmer at Louisiana State University.

Dr. R. G. Bartle of Yale University has been appointed to an assistant professorship at the University of Illinois.

Dr. J. D. Baum of Oberlin College has been promoted to an assistant professorship.

Mr. T. H. Bedwell, previously an instructor at Southern State Teachers College, Springfield, South Dakota, has been appointed to an instructorship at California State Polytechnic College.

Mr. R. J. Beeber, formerly an instructor at St. Peter's College, Jersey City, New Jersey, is now a mathematician with the I. B. M. Corporation, New York City.

Dr. P. R. Beesack, previously a research assistant at Washington University, has been appointed to an assistant professorship at McMaster University.

Dr. Donald C. Benson of Princeton University has been appointed to an assistant professorship at Carnegie Institute of Technology.

Mr. W. H. Benson, retired captain of the United States Navy, has been appointed to an assistant professorship at Dickinson College.

Mr. H. S. Berg, previously an instructor at State Teachers College, St. Cloud, Minnesota, has taken a position as a mathematical analyst at Lockheed Aircraft Corporation, Burbank, California.

Mr. H. H. Berry, formerly a numerical analyst at General Electric Company, Cincinnati, Ohio, has a position as a mathematician at the Avco Manufacturing Company, Crosley Division, Cincinnati.

Mr. D. W. Blakeslee of San Francisco State College has been promoted to an assistant professorship.

Associate Professor W. J. Blundon of Memorial University of Newfoundland is on sabbatical leave at University College, London, England.

Dr. C. C. Braunschweiger, recently a teaching assistant at the University of Wisconsin, has been appointed to an instructorship at Purdue University.

Mr. E. B. Bridgforth, previously a statistician in the Nutrition Division of Vanderbilt Medical School, has been appointed Assistant Professor of Biostatistics, School of Medicine, Vanderbilt University.

Dr. W. E. Briggs, recently a research assistant at the University of Colorado, has been appointed to an instructorship at the University.

Professor S. K. Bright of Austin Peay State College has been appointed Research Scientist at the University of Texas.

Mr. R. C. Brown, Jr., of Glenville State College, West Virginia, has a position as a senior aerophysics engineer at Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas.

Assistant Professor W. P. Brown of Michigan State University has been appointed to an assistant professorship at the University of Toronto.

Mr. R. G. Buschman, formerly a fellow at the University of Colorado, has been appointed to an instructorship at the University.

Mr. C. M. Callahan of Florida State University is now a mathematician at the U. S. Navy Mine Defense Laboratory, Panama City, Florida.

Mr. L. N. Caplan, formerly a student at Carnegie Institute of Technology, has a position as a numerical analyst for the General Electric Company, Evendale, Ohio.

Mr. J. A. Carpenter, previously a mathematician with the Ultrasonic Corporation, Cambridge, Massachusetts, has a position as an engineer for Melpar, Inc., Cambridge.

Miss Margaret J. Cotter, recently a teacher at West Orange High School, New Jersey, is now a mathematics teacher at Dwight Morrow High School, Englewood, New Jersey.

Mr. D. C. Davis, formerly a graduate assistant at the University of Oklahoma, has a position as a development engineer at McDonnell Aircraft Corporation, St. Louis, Missouri.

Dr. Ruth M. Davis of the University of Maryland has accepted a position as a mathematician with the Applied Mathematics Laboratory, David Taylor Model Basin, Carderock, Maryland.

Dr. R. A. Dean of California Institute of Technology has been promoted to an assistant professorship.

Mr. E. J. Delate, previously a group leader in the Yerkes Film Plant, E. I. DuPont de Nemours, Buffalo, New York, has been promoted to the position of Statistical Control Supervisor.

Mr. A. W. Dickinson, formerly a student at the University of Massachusetts, is now a National Science Foundation Fellow at the University of North Carolina.

Mr. J. O. Distad, recently a graduate student at Montana State College,

has been appointed to an instructorship at the University of Alaska.

Assistant Professor L. J. Dixon of Arkansas State College has been promoted to an associate professorship.

Mr. D. L. Dunning, previously an assistant actuary at Warner-Watson Inc., Chicago, Illinois, is now teaching at Elmhurst Junior High School, Illinois.

Professor L. A. Dye of the Citadel has been appointed Head of the Department of Mathematics.

Professor Emeritus W. E. Edington of DePauw University has been appointed Margaret Pilcher Visiting Professor of Mathematics at Coe College for the year 1955–1956.

Professor D. O. Ellis of the University of Florida has accepted a position as associate mathematician with the Rand Corporation, Santa Monica, California.

Mr. R. C. Foster of Tri-State College has a position as an associate engineer at Lockheed Aircraft Corporation, Burbank, California.

Assistant Professor Cleota G. Fry of Purdue University has been promoted to an associate professorship.

Dr. W. M. Gilbert of the State College of Washington has been appointed to an assistant professorship at Iowa State College.

Mr. R. L. Greene, formerly a design engineer with Westinghouse Electric Corporation, Sharon, Pennsylvania, has a position as a design engineer at Chicago Standard Transformer, Chicago, Illinois.

Dr. H. C. Griffith of the University of Connecticut has been appointed to an assistant professorship at Florida State University.

Dr. J. E. Hafstrom of the University of Minnesota, Duluth Branch, has been promoted to an assistant professorship.

Professor N. A. Hall of the University of Minnesota has been appointed Assistant Dean of the Graduate Division of the College of Engineering, New York University.

Mr. S. S. Holland, Jr., previously a mathematician with Army Chemical Corps, Army Chemical Center, Maryland, has a position as a mathematician at Technical Operations Inc., Arlington, Massachusetts.

Mr. R. H. Hoskins, recently actuarial associate with John Hancock Mutual Life Insurance Company, Boston, Massachusetts, has been appointed Assistant Group Actuary of the Company.

Assistant Professor A. V. Houghton, III, of Bradley University has been promoted to an associate professorship.

Dr. J. L. Howell of the University of Delaware has been appointed to an assistant professorship at the University of Alabama.

Dr. P. F. Hultquist of the University of Colorado has been promoted to an assistant professorship.

Assistant Professor T. A. Jeeves of the University of California has a position as a research mathematician for Westinghouse Research Laboratories, East Pittsburgh, Pennsylvania.

Professor Mark Kac of Cornell University is on leave during the current academic year on an Air Force Contract in Geneva, Switzerland.

Associate Professor J. W. Kaiser of Kent State University has been promoted to a professorship.

Assistant Professor Karlis Kaufmanis of Gustavus Adolphus College has been promoted to an associate professorship.

Mr. R. B. Kelman, formerly a student at the University of California, is now a computing analyst for North American Aviation, Downey, California.

Mr. R. R. Kemp, previously a teaching assistant at Massachusetts Institute of Technology, has been promoted to an instructorship at the Institute.

Mr. Hewitt Kenyon of the University of Rochester has been promoted to an assistant professorship.

Mr. E. G. Kimme, formerly a teaching assistant at the University of Minnesota, has been appointed to an instructorship at Oregon State College.

Associate Professor V. L. Klee, Jr., of the University of Washington has been appointed Visiting Associate Professor at the University of California at Los Angeles for the academic year 1955–1956.

Assistant Professor L. A. Kokoris of the University of Washington has been appointed to an assistant professorship at Washington University.

Mr. H. P. Kuang, formerly a research fellow and senior statistician at the University of Minnesota, has been appointed to a professorship at Agricultural and Technical College, Greensboro, North Carolina.

Mr. Y. L. Luke, research mathematician at Midwest Research Institute, Kansas City, Missouri, has been promoted to the position of Head of the Mathematical Analysis Section.

Dr. M. D. Marcus of the University of British Columbia has been promoted to an assistant professorship.

Mr. P. L. Marshall has a position as a project aerodynamicist with North American Aviation Corporation, Columbus, Ohio.

Mr. H. J. McBlaine, Jr., formerly a training instructor in meteorology, Department of Weather School, Chanute Air Force Base, Illinois, is now a mathematician at the U. S. Naval Ordnance Laboratory, Corona, California.

Associate Professor S. S. McNeary of Drexel Institute of Technology has been promoted to a professorship.

Assistant Professor D. M. Merriell of Robert College, Istanbul, Turkey, has been promoted to an associate professorship.

Professor W. E. Milne of Oregon State College has retired.

Associate Professor T. W. Moore of the United States Naval Academy has been promoted to a professorship.

Mr. T. P. Mulhern, previously a graduate student at Brown University, has been appointed to a research assistantship at the University.

Dr. D. J. Myatt, a research mechanical engineer with the Atlantic Research Corporation, Alexandria, Virginia, has been appointed to an associate professorship at Antioch College.

Associate Professor Henry Parkus of Michigan State University has been appointed to a professorship at the University of Technology in Vienna.

Mr. K. C. Peng, previously a statistical quality control engineer at Parke, Davis and Company, Detroit, Michigan, has a position as a mathematical statistician with the Chrysler Corporation, Detroit.

Associate Professor Ruth M. Peters of St. Lawrence University has been promoted to a professorship.

Mr. C. W. Pflaum, formerly an instructor at John Marshall High School, Rochester, New York, has been appointed Head of the Department of Mathematics of the School.

Professor Wladimir Seidel of the University of Rochester has been appointed to a professorship at the University of Notre Dame.

Dr. Harry Siller, previously a statistician for the Jewish Education Committee, New York City, has been appointed to an assistant professorship at Hofstra College.

Professor W. S. Snyder of the University of Tennessee has been appointed to the staff of the Oak Ridge National Laboratory, Oak Ridge, Tennessee.

Mr. W. K. Spears has accepted a position as engineer with the Electronics Corporation of America, Cambridge, Massachusetts.

Mr. V. E. Thomas, formerly an instructor at West Virginia University, has a position as a performance analyst for Pratt and Whitney Aircraft, East Hartford, Connecticut.

Assistant Dean C. W. Trigg of Los Angeles City College has been promoted to the position of Dean of Instruction; Mr. Trigg also served as Dean of the Summer Session of the College.

Assistant Professor J. M. Wolfe of Brooklyn College has been promoted to an associate professorship.

Mr. T. J. Wolinski, previously a student at St. John's College, Brooklyn, New York, has a position as a mathematician at the Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland.

Professor R. C. Yates of Virginia Polytechnic Institute has been appointed Professor and Head of the Department of Mathematics of the College of William and Mary.

Mr. F. R. Yett of Iowa State College has been appointed to an assistant professorship at Long Beach State College.

Mr. J. A. Zilber of Johns Hopkins University has been appointed to an instructorship at the University of Illinois.

Professor M. A. Girshick of Stanford University died on March 2, 1955.

Mr. George Hartnell, formerly a research geologist with the U. S. Coast and Geodetic Survey, died on March 20, 1955. He was a member of the Association for thirty years.

Professor H. C. Shaub, head of the Department of Mathematics of Washington and Jefferson College, died on July 20, 1955. He was a member of the As-

sociation for thirty-four years.

Professor Augustus Sisk of Maryville College died on February 1, 1955. He was a member of the Association for twenty years.

Professor Emeritus E. J. Townsend, formerly dean of the College of Science, University of Illinois, died on July 8, 1955.

Mr. E. L. Vanderburgh of Pueblo Junior College, Colorado, died on October 27, 1954.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-SIXTH SUMMER MEETING OF THE ASSOCIATION

The thirty-sixth summer meeting of the Mathematical Association of America was held at the University of Michigan, Ann Arbor, Michigan, on Monday and Tuesday, August 29 and 30, 1955, in conjunction with the summer meetings of the American Mathematical Society, the Association for Symbolic Logic, the Econometric Society, the Industrial Mathematics Society, the Institute of Mathematical Statistics, the Pi Mu Epsilon Fraternity, and the Society for Industrial and Applied Mathematics. Eleven hundred and seventy persons were registered, including the following four hundred and eighty-nine members of the Association:

Smbat Abian, R. P. Agnew, M. I. Aissen, A. A. Albert, H. W. Alexander, Bess E. Allen, W. R. Allen, C. B. Allendoerfer, A. G. Anderson, R. V. Andree, H. A. Antosiewicz, K. J. Arnold, R. A. Askey, W. L. Ayres, J. L. Bagg, R. W. Bagley, R. P. Bailey, Frances E. Baker, J. M. Barbour, R. H. Bardell, Joshua Barlaz, R. W. Barnard, W. E. Barnes, I. A. Barnett, D. Y. Barrer, J. H Barrett, R. C. F. Bartels, M. A. Basoco, P. T. Bateman, M. T. Battles, Jr., J. D. Baum, H. M. Beatty, W. A. Beck, E. G. Begle, Louise G. Belai, Richard Bellman, J. S. Bendat, H. A. Bender, Dean C. Benson, Dorothy L. Bernstein, H. H. Berry, H. R. Beveridge, W. S. Bicknell, Kurt Bing, R. H. Bing, C. J. Blackall, D. W. Blackett, Shirley A. Blackett, David B! ckwell, R. L. Blair, J. H. Blau, H. D. Block, G. M. Bloom, M. Isobel Blyth, R. D. Boswell, Jr., S. G. Bourne, Julia W. Bower, C. B. Boyer, J. W. Brace, Fred Brafman, G. U. Brauer, C. C. Braunschweiger, C. S. Brewster, C. F. Briggs, H. W. Brinkmann, Paul Brock, Arlen Brown, Marjorie L. Browne, R. H. Bruck, R. C. Buck, G. C. Byers, J. M. Calloway, E. A. Cameron, E. J. Camp, H. H. Campaigne, H. E. Campbell, C. E. Capel, Dorothy I. Carpenter, R. E. Carr, W. B. Carver, Jeremiah Certaine, Lamberto Cesari, C. Y. Chao, S. S. Chern, P. L. Chessin, R. V. Churchill, A. G. Clark, E. H. Clarke, Nathaniel Coburn, C. J. Coe, L. M. Coffin, L. W. Cohen, A. J. Coleman, E. P. Coleman, N. B. Conkwright, C. H. Cook, T. F. Cope, A. H. Copeland, Sr., R. R. Coveyou, W. H. H. Cowles, V. F. Cowling, H. S. M. Coxeter, C. C. Craig, E. H. Crisler, J. W. Crispin, Jr., Helen F. Cullen, A. B. Cunningham, H. B. Curry, J. H. Curtiss, John F. Daly, D. A. Darling, Robert Davies, R. L. Davis, Ruth M. Davis, Violet B. Davis, D. E. Deal, E. R. Deal, R. B. Deal, Jr., R. A. Dean, R. Y. Dean, M. L. DeMoss, E. K. Dorff, Jim Douglas, Jr., H. H. Downing, W. L. Duren, Jr., P. S. Dwyer, J. C. Eaves, W. F. Eberlein, Albert Edrei, Carolyn Eisele, J. D. Elder, J. G. Elliott,

D. O. Ellis, Wade Ellis, M. P. Emerson, M. P. Epstein, H. P. Evans, H. S. Everett, W. H. Fagerstrom, Catherine S. Feeley, W. J. Feeney, H. F. Fehr, F. A. Ficken, J. V. Finch, N. J. Fine, D. T. Finkbeiner, II, C. H. Fischer, N. C. Fisk, G. E. Forsythe, M. K. Fort, Jr., Tomlinson Fort, J. S. Frame, Evelyn Frank, C. H. Frick, C. V. Fronabarger, T. C. Fry, R. E. Fullerton, J. W. Gaddum, T. M. Gallie, Jr., R. A. Gambill, G. N. Garrison, H. M. Gehman, K. G. Getman, B. P. Gill, Leonard Gillman, Wallace Givens, A. M. Gleason, Casper Goffman, Michael Goldberg, Samuel Goldberg, Michael Golomb, W. A. Golomski, D. B. Goodner, R. M. Gordon, S. H. Gould, Arthur Grad, L. M. Graves, R. L. Graves, J. W. Green, F. L. Griffin, J. S. Griffin, Jr., G. W. Grotts, V. G. Grove, W. E. Grove, P. E. Guenther, H. M. Gurk, Theodore Hailperin, Franklin Haimo, Marshall Hall, Jr., P. R. Halmos, J. W. Hamblen, P. C. Hammer, H. W. Handsfield, Frank Harary, W. J. Hardell, W. L. Hart, H. L. Harter, M. C. Hartley, T. W. Hatcher, G. E. Hay, Katharine E. Hazard, T. J. Head, H. S. Heaps, G. A. Hedlund, E. R. Heineman, R. G. Helsel, Melvin Henriksen, Edwin Hewitt, T. H. Hildebrandt, T. W. Hildebrandt, J. T. Hinely, Jr., J. J. L. Hinrichsen, A. J. Hoffman, Roslyn B. Hoffman, S. P. Hoffman, Jr., Walter Hoffman, R. V. Hogg, F. E. Hohn, L. Aileen Hostinsky, D. B. Houghton, A. S. Householder, C. C. Hsiung, R. C. Huffer, M. Gweneth Humphreys, W. R. Hutcherson, Jane C. Ingersoll, H. G. Jacob, Jr., Bernard Jacobson, L. A. Jehn, F. I. John, L. W. Johnson, R. E. Johnson, A. W. Jones, F. B. Jones, H. T. Jones, L. O. Jones, P. S. Jones, Mark Kac, L. H. Kanter, Wilfred Kaplan, Irving Kaplansky, L. C. Karpinski, Chosaburo Kato, Leo Katz, M. W. Keller, C. E. Kelley, J. E. Kelley, L. M. Kells, J. G. Kemeny, J. H. B. Kemperman, D. E. Kibbey, E. C. Kiefer, S. H. Kimball, M. S. Klamkin, L. A. Knowler, R. J. Koch, F. W. Kokomoor, L. A. Kokoris, Jacob Korevaar, R. R. Korfhage, D. M. Krabill, Max Kramer, G. R. Kuhn, Harold W. Kuhn, Solomon Kullback, L. C. Labowitz, R. J. Lambert, R. E. Langer, E. H. Larguier, J. P. LaSalle, C. G. Latimer, E. B. Leach, J. R. Lee, Marguerite Lehr, R. A. Leibler, Walter Leighton, K. B. Leisenring, F. C. Leone, W. J. Leveque, Norma L. Lindemann, B. W. Lindgren, A. E. Livingston, B. J. Lockhart, K. L. Loewen, F. W. Lott, Jr., D. B. Lowdenslager, L. L. Lowenstein, C. I. Lubin, Y. L. Luke, R. W. MacDowell, W. G. Madow, C. G. Maple, Morris Marden, A. M. Mark, A. D. Martin, N. M. Martin, Ceslovas Masaitis, J. C. Mathews, K. O. May, B. H. McCandless, P. J. McCarthy, N. H. McCoy, S. W. McCuskey, W. C. McDaniel, Edith A. McDougle, W. R. McEwen, A. W. McGaughey, J. H. McKay, J. E. McLaughlin, R. M. McLeod, A. E. Meder, Jr., L. E. Mehlenbacher, Paul Meier, D. F. Mela, E. P. Merkes, B. E. Meserve, D. M. Mesner, Fred Meyer, D. D. Miller, E. E. Moise, Harriet F. Montague, Mabel D. Montgomery, J. T. Moore, W. K. Moore, Vera T. Morris, F. C. Mosteller, L. T. Moston, E. J. Moulton, H. T. Muhly, S. B. Myers, W. H. Myers, H. W. Nace, H. E. Nelson, E. D. Nering, Morris Newman, E. A. Nordhaus, E. S. Northam, E. P. Northrop, C. O. Oakley, E. N. Oberg, R. H. Oehmke, Theresa M. C. Oehmke, Rufus Oldenburger, E. G. Olds, L. F. Ollman, C. E. Olsen, Joseph Oppenheim, G. M. Ortner, W. R. Orton, Jr., Morris Ostrofsky, F. F. Otis, R. R. Otter, J. C. Oxtoby, L. J. Paige, O. O. Pardee, G. P. Paternoster, G. A. Paxson, W. H. Pell, F. W. Perkins, H. P. Pettit, C. W. Pflaum, C. R. Phelps, George Piranian, Everett Pitcher, J. C. Polley, G. B. Price, K. S. Purdie, A. L. Putnam, J. W. Querry, Gustave Rabson, Tibor Rado, Henry Rainbow, G. Y. Rainich, E. D. Rainville, Gordon Raisbeck, G. E. Raynor, C. B. Read, M. O. Reade, W. T. Reid, Irving Reiner, Haim Reingold, C. E. Rhodes, D. E. Richmond, P. R. Rider, J. D. Riley, R. F. Rinehart, J. M. Robb, G. deB. Robinson, Robin Robinson, Selby Robinson, F. Virginia Rohde, M. I. Rose, Saul Rosen, Alex Rosenberg, P. C. Rosenbloom, Arthur Rosenthal, A. E. Ross, E. H. Rothe, L. J. Rouse, Mary E. Rudin, H. J. Ryser, Hans Samelson, W. C. Sangren, A. C. Schaeffer, Alice T. Schafer, R. D. Schafer, Robert Schatten, Henry Scheffe, E. D. Schell, Edith R. Schneckenburger, J. J. Schoderbek, Pincus Schub, B. L. Schwartz, W. T. Scott, J. G. Sharp, A. L. Shields, Annette Sinclair, David Singer, James Singer, Sister Mary Carmel, M. F. Smiley, H. W. Smith, Ernst Snapper, E. J. Specht, G. L. Spencer, II, Abraham Spitzbart, J. R. Stallings, Jr., Maria W. Steinberg, H. E. Stelson, C. F. Stephens, Rothwell Stephens, A. D. Stewart, B. M. Stewart, Ruth W. Stokes, R. R. Stoll, Irwin Stoner, W. L. Strother, Anna K. Suter, R. L. Swain, W. C. Swift, T. T. Tanimoto, O. E. Taulbee, William Clare Taylor, H. P. Thielman, G. B. Thomas, Jr., G. L. Thompson, D. E. Thoro, R. M. Thrall, W. J. Thron, H. S. Thurston, John Todd, M. L. Tomber, C. B. Tompkins, Eugenia I. Trapp, C. K. Tsao, A. W. Tucker, J. W. Tukey, J. B. Tysver,

F. E. Ulrich, W. R. Utz, Jr., Henry Van Engen, Helen E. Van Sant, T. L. Wade, Jr., R. W. Wagner, E. A. Walker, R. J. Walker, A. D. Wallace, S. E. Warschawski, G. C. Webber, J. V. Wehausen, W. G. Weideman, E. T. Welmers, F. J. Weyl, Frederick R. White, P. M. Whitman, L. R. Wilcox, Marie S. Wilcox, R. L. Wilder, S. S. Wilks, R. L. Wilson, G. N. Wollan, F. L. Wren, F. B. Wright, Jr., C. T. Yang, J. E. Yarnelle, J. W. T. Youngs, J. L. Zemmer, Jr., J. A. Zilber, R. E. Zindler.

Sessions of the Association were held on Monday morning and afternoon in the Natural Science Auditorium of the University of Michigan with President W. L. Duren and Vice-President H. S. M. Coxeter presiding. The Session on Tuesday morning was held in the Auditorium of Rackham Hall with Vice-President G. B. Price and President Duren presiding. The fourth series of Earle Raymond Hedrick Lectures was delivered by Professor Mark Kac of Cornell University. The Program Committee for the meeting consisted of J. W. T. Youngs, Chairman; R. H. Bing, and V. L. Klee, Jr.

FIRST SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Familiar Things from an Unfamiliar Point of View"; Lecture I, "From Vieta to Foundations of Modern Probability Theory," by Professor Mark Kac, Cornell University.

Mathematics for Social Scientists, II (Joint Session with the Econometric Society). Panel Chairman: Professor R. M. Thrall, University of Michigan. Panel Members: Professor J. G. Kemeny, Dartmouth College, Professor W. G. Madow, University of Illinois and Stanford University, Professor Frederick Mosteller, Harvard University, Professor C. O. Oakley, Haverford College.

SECOND SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Familiar Things from an Unfamiliar Point of View"; Lecture II, "From Vieta to Foundations of Modern Probability Theory," by Professor Mark Kac.

"Mathematics as a Profession Today": "In the university," by Professor G. B. Price, University of Kansas; "In government," by Dr. R. A. Leibler, Department of Defense; "In industry," by Dr. T. C. Fry, Bell Telephone Laboratories.

THIRD SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Familiar Things from an Unfamiliar Point of View"; Lecture III, "From Kinetic Theory to Continued Fractions," by Professor Mark Kac.

Business Meeting of the Association.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Monday evening in the Michigan Union. Twenty-five of the forty-one members of the Board were present. Among the more important items of business transacted were the following:

President Duren announced the appointment of a Committee on Mathematical Personnel and Education consisting of Tomlinson Fort, Chairman, E. A. Cameron, H. P. Evans, M. Gweneth Humphreys, Ivan Niven, E. P. Northrop, and E. T. Welmers.

It was also announced that G. B. Price had agreed to serve as chairman of the Committee to Study the Activities of the Association.

It was voted to authorize the publication of "Irrational Numbers" by Ivan Niven as Carus Monograph No. 11.

The Board voted to approve the action of the Putnam Trustees in increasing from $1500 to $2500 the value of the scholarship at Harvard University or at Radcliffe College which is awarded annually to one of the five highest individual contestants in the Putnam Mathematical Competition.

BUSINESS MEETING OF THE ASSOCIATION

At a business meeting of the Association held on Tuesday morning, announcement was made of a grant of $25,000 received by the Association from the Ford Foundation for the work of the Committee on the Undergraduate Program and a grant of $20,500 from the National Science Foundation for the continued support of the Program of Visiting Lecturers.

Professor D. E. Richmond reported for the Committee on Visiting Lecturers; Professor J. S. Frame reported on the Employment Register; Professor W. L. Duren reported for the Committee on the Undergraduate Program; and Professor A. W. Tucker told of the newly appointed Commission on Mathematics of the College Entrance Examination Board.

The Secretary announced that the membership of the Association was 5999 on August 25.

MEETING OF SECTION OFFICERS

A meeting of officers of the Sections of the Association was held on Tuesday evening in the Michigan Union. Thirty-six persons were present representing twenty-one Sections and the New England Region.

Among the topics discussed were contests among high school students sponsored by sections, programs of section meetings, and the financial support of the sections. It was agreed that these meetings of section officers serve a useful purpose and it was voted to continue to hold such meetings in connection with the summer meetings of the Association.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Tuesday morning through Friday afternoon. The colloquium lectures on "Jordan Algebras" were delivered by Professor Nathan Jacobson of Yale University. Invited addresses were given by Professors Raoul Bott and Edwin Hewitt. The Econometric Society held its meetings from Monday through Thursday. The Pi Mu Epsilon Fraternity met on Tuesday. The Society for Industrial and Applied Mathematics

met on Tuesday and Wednesday. The Institute of Mathematical Statistics met from Tuesday through Friday. The Association for Symbolic Logic met on Thursday and the Industrial Mathematics Society met on Friday.

The Committee on the Mathematical Training of Social Scientists held a conference on Saturday and Sunday, August 27 and 28. The Commission on Mathematics of the College Entrance Examination Board met on these same dates.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of: Wilfred Kaplan, Chairman; P. S. Dwyer, H. M. Gehman, P. S. Jones, W. J. LeVeque, R. M. Thrall, R. L. Wilder, J. W. T. Youngs.

Registration headquarters was in the lobby of the West Quadrangle Dormitories of the University of Michigan. Dormitory accommodations and meals were available in West Quadrangle.

A tea in Inglis House for the women attending the meeting was held on Tuesday afternoon, and a square dance on Tuesday evening. A guided tour of the campus was scheduled for Wednesday morning, a picnic at Kensington Metropolitan Park for Wednesday afternoon and evening, a coffee hour for the ladies on Thursday morning and a concert for Thursday evening.

A supervised play program for children was provided throughout the week at the University Elementary School and free baby-sitting service was provided from Monday through Thursday evenings.

The Employment Register, listing positions available for mathematicians, was open for inspection during the week in Mason Hall.

On Tuesday morning, Vice-President M. L. Niehuss of the University of Michigan greeted the assembled mathematicians on behalf of the University. Later, Professor Harold W. Kuhn presented a motion on behalf of the eight organizations meeting at the University of Michigan expressing "thanks and appreciation, especially to the local members of the Committee on Arrangements who, in spite of the unusually large attendance, have provided exceptionally complete accommodations and services to make the week in Ann Arbor a memorable one for us and for our families."

HARRY M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 54 persons have been elected to membership by the Board of Governors on applications duly certified.

ARLYNE G. ADELSTEIN, M.A. (Western Reserve) University Heights, Ohio.

J. H. BAILEY, Student, University of Rhode Island.

MAJ. J. F. BLACKBURN, Ph.D. (North Carolina) Asst. Professor, U. S. Air Force Academy, Denver, Colo.

C. K. BRADSHAW, M.S. (Nevada) Instr., University of Nevada.

J. P. BRANNEN, B.S. (A.&M.C. of Texas) Instr., Sweeny High School, Texas.

F. C. BREISCH, A.B. in Ed. (Michigan) Whitmore Lake, Michigan.

R. K. BUTNER, Ph.D. (S.U. of Iowa) Asst. Professor, Ohio University.

H. L. CARLSON, M.S. (Wisconsin) Electronic Applications Specialist, National Cash Register Co., Dayton, O.

M. M. CHIRICO, M.A. (Syracuse) Mathematician, Aberdeen Proving Ground, Md.

L. C. DEAN, JR., M.S. (North Texas S.C.) Grad. Asst., Iowa State College.

M. R. DEMERS, M.A. (Buffalo) Asst. Professor, University of Nevada.

C. W. DYCHE, B.A. (Linfield) Grad. Asst., Oregon State College.

MRS. HAZEL E. EVANS, B.A. (Allegheny) Lecturer, University of Pittsburgh.

MILTON FELSTEIN, B.A. (Texas) Fire Control Design Engr., U. S. Naval Gun Factory, Washington, D. C.

D. JOAN GALLAGHER, B.A. (Colorado) Kimball, South Dakota.

G. H. F. GARDNER, Ph.D. (Princeton) Instr., Cornell University.

C. D. GORMAN, A.B. (Missouri) Asst. Instr., University of Missouri.

E. H. GROSSMAN, Vancouver, B. C., Canada.

F. L. HARDY, A.B. (Oglethorpe) Graduate School, Emory University.

ENID B. JACOBSON, M.A. (Temple) Instr., Temple University.

C. E. JAEGER, M.A. (Claremont Grad. School) Instr., Chaffey High School, Ontario, Calif.

F. I. JOHN, M.A. (Boston U.) Pfc., United States Army.

J. A. KALMAN, Ph.D. (Harvard) University College, Auckland, New Zealand.

MR. MYUNGHWAN KIM, Student, University of Alabama.

H. G. LANGE, Chicago, Illinois.

R. P. LANGLANDS, Student, University of British Columbia.

R. A. LUEBBE, M.A. (Cincinnati) Grad. Student, University of Cincinnati.

Y. L. LUKE, M.S. (Illinois) Head, Mathematical Analysis Section, Midwest Research Institute, Kansas City, Mo.

JEANNETTE S. MAGASINY, Student, Temple University.

PAUL MANOS, Student, Western Reserve University.

W. C. MASON, Student, Massachusetts Institute of Technology.

J. C. MATHEWS, Student, Iowa State College.

MRS. DOROTHY L. B. MCCULLOUGH, Student, Baylor University.

R. A. MCHAFFEY, B. S. (Brooklyn P.I.) Grad. Asst., Iowa State College.

G. E. NEU, B.A. (Houghton) Instr., University of Buffalo.

M. G. OSSESIA, B.S. (Pittsburgh) Grad. Student, University of Pittsburgh.

R. V. PARKER, Teacher, Scole School, Diss, Norfolk, England.

F. R. POOLE, B.S. (Nebraska) Instr., Ohio University.

R. M. RILEY, M.S. (Minnesota) Member of the Technical Staff, Bell Telephone Labs., Murray Hill, N.J.

SOLOMON RUBINSTEIN, Student, City College of the City of New York.

F. D. RUDOLPH, JR., B.A. (Williams) Field Representative, Addison-Wesley Publishing Co., Cambridge, Mass.

W. H. SAYRE, Student, Columbia University.

R. M. SHANK, B.A. (Marshall) Grad. Student, University of Kentucky.

BROCK SHUGART, B.S. (Texas) Division Engr., Pacific Valves, Houston, Texas.

J. J. STIPANOWICH, M.S. (Illinois) Asst. Professor, Western Illinois State College.

JANE M. SUNDGAARD, M.A. (Minnesota) Grad. Student, University of Minnesota.

G. M. TRUE, B.A. (Williams) Field Representative, Addison-Wesley Publishing Co., Cambridge, Mass.

J. F. ULRICH, M.A. (Chicago) Teacher, Arlington Heights Township High School, Ill.

S. P. VESELKA, M.S. (N.Y.S.C. for Teachers, Albany) United States Army.

JANET M. WEBER, B.S. (Illinois Wesleyan) Instr., Township High School, Sterling, Ill.

ELIZABETH A. WELLS, B.A. (Phillips) Grad. Asst., University of Oklahoma.

DAVID WERNER, M.S. (Michigan) Mathematician, Naval Proving Ground, Dahlgren, Va.

J. J. WESOLOWSKI, B.S. (St. Mary's) Grad. Student, St. Mary's University.

T. M. WILM, Student, St. John's College, Brooklyn, N. Y.

THE MARCH MEETING OF THE KANSAS SECTION

The fortieth annual meeting of the Kansas Section of the Mathematical Association of America was held at Fort Hays State College, Hays, Kansas, on March 26, 1955. Professor E. C. Stopher, Chairman of the Section, presided.

There were 65 in attendance, including the following 13 members of the Association:

Paul Eberhart, W. C. Foreman, R. W. Gibson, Laura Z. Greene, V. D. Nyhoff, G. B. Price, C. B. Read, L. M. Reagan, Sister M. Nicholas, E. C. Stopher, Robert H. Thompson, Wilmont Toalson, Ferna E. Wrestler.

The following officers were elected for the year 1955–1956: Chairman, Professor Ferna E. Wrestler, University of Wichita; Vice-Chairman, Professor W. R. Scott, University of Kansas; Secretary-Treasurer, Professor Laura Z. Greene, Washburn University of Topeka.

The following papers were presented at the morning and afternoon sessions:

1. *Empirical formula for survival*, by Mr. L. W. Whiteside, Boeing Airplane Company, Wichita, introduced by the Chairman.

2. *Analysis of certain special power forms*, by Mr. R. D. Sinkhorn, University of Wichita.

The paper consisted of an analysis of infinite continued powers with special emphasis on forms in which the base and each of the individual exponents were equal. It was shown that such a power exists for a positive base if and only if this base is less than or at most equal to e; in the case of the zero base, the power oscillates between zero and unity. The cases where the exponents were unlike, negative, or complex were not discussed.

3. *Magic circles*, by Sister M. Nicholas, Marymount College.

There are magic circles as well as magic squares. Magic circles are n concentric circles cut by n diameters with numbers arranged on their intersections in such a way that the sums on all the circles and diameters are equal.

We find mention of magic circles and magic squares in the early Chinese and Japanese books on arithmetic. They placed the 1 in the center of the circles and did not use it in their sums. We find that the sums for this arrangement are equal to $2n^3+3n$, where n is the number of circles and diameters. A second arrangement can be made using 1. The sums for this second arrangement are equal to $2n^3+n$.

Another type of magic circles is a set of interesting circles with numbers placed on the points of intersection of the circles. The numbers on each circle give the same sum.

4. *Some remarks on distance geometries*, by Professor W. L. Stamey, Kansas State College, by title.

5. *A modified course in college geometry*, by Professor J. D. Haggard, Kansas State Teachers College, by title.

6. *A problem of an artist*, by Professor G. W. Smith, University of Kansas, by title.

LAURA Z. GREENE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-ninth Annual Meeting, Rice Institute, Houston. Texas, December 30, 1955.

Thirty-seventh Summer Meeting, University of Washington, Seattle, Washington, August 20–21, 1956.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Geneva College, Beaver Falls, Pennsylvania, Spring, 1956.

ILLINOIS, Eastern Illinois State College, Charleston, May 11-12, 1956.

INDIANA, Wabash College, Crawfordsville, May 5, 1956.

IOWA, Grinnell College, Grinnell, April 20—21, 1956.

KANSAS, University of Wichita, April 21, 1956.

KENTUCKY

LOUISIANA-MISSISSIPPI, McNeese State College, Lake Charles, Louisiana, February 17-18, 1956.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Catholic University, Washington, D. C., December 3, 1955.

METROPOLITAN NEW YORK, Stevens Institute of Technology, Hoboken, New Jersey, April 28, 1956.

MICHIGAN, University of Michigan, Ann Arbor, March, 1956.

MINNESOTA

MISSOURI, Fontbonne College, St. Louis, Spring, 1956.

NEBRASKA, University of Nebraska, Linclon, April 21, 1956.

NEW ENGLAND, Organizational Meeting, University of New Hampshire, Durham, November 26, 1955.

NORTHERN CALIFORNIA, Stanford University, Stanford, January 14, 1956.

OHIO, April, 1956.

OKLAHOMA

PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.

PHILADELPHIA, University of Pennsylvania, Philadelphia, November 26, 1955.

ROCKY MOUNTAIN, University of Utah, Salt Lake City, May 4-5, 1956

SOUTHEASTERN, University of Georgia, Athens, March 16–17, 1956.

SOUTHERN CALIFORNIA, Pomona College, Claremont, March 17, 1956.

SOUTHWESTERN, New Mexico College of Agriculture and Mechanical Arts, Las Cruces, April, 1956.

TEXAS, Southwest Texas State Teachers College, San Marcos, April, 1956.

UPPER NEW YORK STATE, Alfred University, Alfred, April 28, 1956.

WISCONSIN, Marquette University, Milwaukee, May, 1956.

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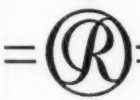

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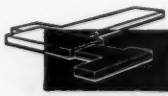

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